Log-Optimality & Multi-Armed Sequential Hypothesis Testing

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Outline

1. What is sequential hypothesis testing (by betting)?
2. How are sequential hypothesis tests derived?
3. Defining and deriving optimal sequential tests.
4. Multi-armed sequential hypothesis testing

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A motivating example to keep in mind: **treatment effect testing**.

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$$H_0 : \text{trt effect} = 0$$
$$H_1 : \text{trt effect} \neq 0$$

$$\alpha := 0.01$$

Step 2: Recruit n patients and randomize (**trt** or **ctrl**).

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No, this is "p-hacking".

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Sequential testing provides one solution to these scenarios.

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(Modern breakthroughs in recent years; textbook by Ramdas & Wang [2025])

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$$\textit{(e.g. } \mathcal{P} = \{P : \textit{trt effect} = 0\} \textit{ versus } \mathcal{Q} = \{P : \textit{trt effect} > 0\}\textit{)}$$

We are tasked with finding a test $\phi_n^{(\alpha)} \equiv \phi^{(\alpha)}(X_1, \ldots, X_n)$ that outputs 1 (rejects $\mathcal{P}$ in favour of $\mathcal{Q}$) with small probability under $\mathcal{P}$.

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Fixed-n test:	$\forall n \in \mathbb{N},\ \sup_{\mathsf{P} \in \mathcal{P}} \mathsf{P}\left(\phi_n^{(\alpha)} \text{ rejects}\right) \leqslant \alpha.$
Sequential test:	$\sup_{\mathsf{P} \in \mathcal{P}} \mathsf{P}\left(\exists n \in \mathbb{N} : \phi_n^{(\alpha)} \text{ rejects}\right) \leqslant \alpha.$

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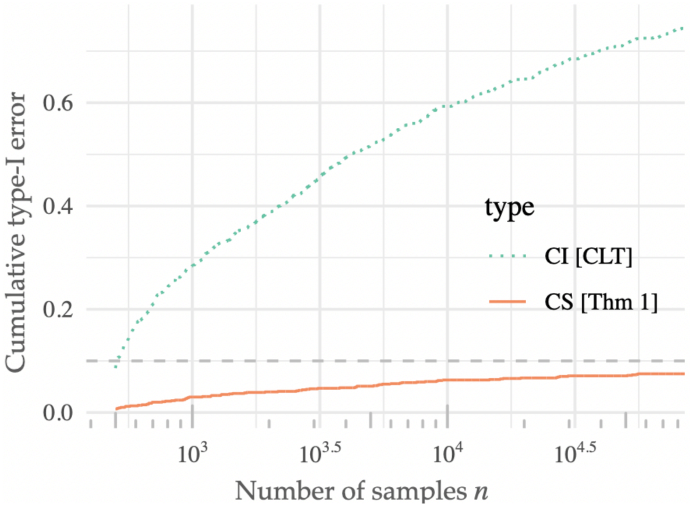

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As a brief aside, note the difference between *asymptotic* (CLT-like) and *non-asymptotic* testing.

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For the **asymptotic, anytime-valid** case, see Robbins–Siegmund [1970], **W-S** et al., and Bibaut–Kallus–Lindon [2022].

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2. Set the test as $\phi_n^{(\alpha)} := \mathbb{1}\{W_n \geqslant 1/\alpha\}$.

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When $x = 1/\alpha$ and $\mathbb{E}_{\mathsf{P}}[W_1] \leqslant 1$ for each $\mathsf{P} \in \mathcal{P}$, we get

$$\forall \alpha \in (0,1), \quad \sup_{\mathsf{P}\in\mathcal{P}} \mathsf{P}\left(\exists n \in \mathbb{N} : \phi_n^{(\alpha)} = 1\right) \leqslant \alpha.$$

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However, lots of progress has been made in recent years.

Example: Testing the mean of a bounded random variables.

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$$W_n := \prod_{i=1}^{n} (1 + \gamma_i \cdot (2X_i - 1))$$

forms a test martingale for any $[0,1]$-valued *predictable* $(\gamma_n)_{n\in\mathbb{N}}$.

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Therefore, $\phi_n^{(\alpha)} := \mathbb{1}\{W_n \geqslant 1/\alpha\}$ yields a sequential test for $\mathcal{P}$.

It is **not** obvious how to choose γ_i... more on that later.

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This is the *only* “good” test martingale (**W-S**&R’24, Clerico [2025]).

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A nonnegative random variable $E \geqslant 0$ is said to be a $\mathcal{P}$-e-value if

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One can re-cast several testing problems *(bounded means, two-sample, independence, equality of bounded tuples, testing randomness online, etc.)* from the literature with convex combinations of carefully chosen e-values.

Throughout, consider the process $(W_n)_{n\in\mathbb{N}}$ given by (predictable) convex combination of $d+1$ e-values:

$$W_n := \prod_{i=1}^{n} \lambda_i^\top \mathbf{E}_i,$$

$$\text{where} \quad \mathbf{E}_i = (E_0, \ldots, E_d) \quad \text{is a tuple of } \mathcal{P}\text{-e-values},$$

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Some other special cases found in the literature

One-sided bounded mean testing: Set $\mathbf{E}_i = (1, X_i/\mu_0)$.
Two-sided bounded mean testing: Set $\mathbf{E}_i = ((1 - X_i)/(1 - \mu_0), X_i/\mu_0)$.
Two-sample testing: Set $\mathbf{E}_i = (1, g^\star(X_i) - g^\star(Y_i))$ for a witness f'n $g^\star$.

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Indeed, $\prod_{i=1}^{n} {\lambda_i}^\top \mathbf{E}_i$ yields a valid sequential test under $\mathcal{P}$ for *any* choice of $(\lambda_n)_{n\in\mathbb{N}}$. Are there "better" choices than others under $\mathcal{Q}$?

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See Gergely Neu [2024]'s discussion of **W-S**&R'24.

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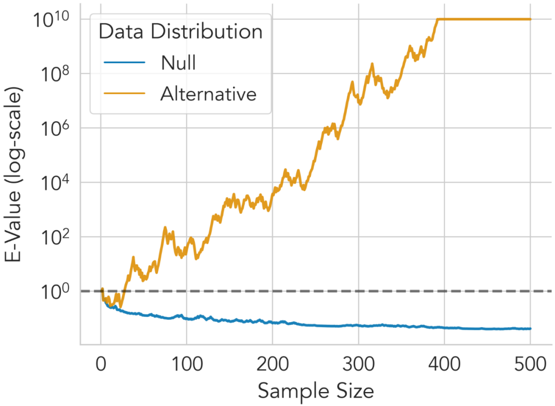
We show that these are optimized via the exact same criterion.

(i) Growth-rate-optimality.

(Kelly ['56], Long Jr. ['90], Karatzas & Kardaras [2007], Grünwald–de Heide–Koolen [2024], Larsson–Ramdas–Ruf [2024])



An e-process is expected to be small under the ***null****;*
we want it to grow large under the ***alternative****.*

Image credit: YJ Choe.

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Observe by the strong law of large numbers:

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W_n &= \exp\left\{ n \cdot \frac{1}{n} \sum_{i=1}^{n} \log\left(\lambda_i{}^\top \mathbf{E}_i \right) \right\} \\
&\approx \exp\left\{ n \cdot \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda_i{}^\top \mathbf{E}_i \right) \right] + o(n) \right\}
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So, if data comes from $\mathsf{Q} \in \mathcal{Q}$, it is reasonable to want to choose λ_i so as to maximize:

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This is the essence of the "Kelly criterion" (J.L. Kelly Jr., [1956]).

Proposition: Upper bound on the asymptotic growth rate

Suppose $\mathbf{E}_1, \mathbf{E}_2, \cdots \sim \mathsf{Q} \in \mathcal{Q}$. For any predictable sequence $(\lambda_n)_{n \in \mathbb{N}}$, we have that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \log\left(\lambda_i{}^\top \mathbf{E}_i\right) \leqslant \max_{\lambda \in \triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log(\lambda^\top \mathbf{E}_1)\right]$$

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One can also motivate the Kelly criterion from a more "purely statistical" perspective.

(ii) Measuring optimality through expected rejection times

(Wald [1945], Breiman [1961], Honda, Takemura, Kaufmann, Agrawal, Juneja, Koolen, and several others in recent years)

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Proposition: Lower bound on the expected stopping time

For any choice of $(\lambda_n)_{n\in\mathbb{N}}$, let τ_α be the resulting first hitting time of $W_n := \prod_{i=1}^n {\lambda_i}^\top \mathbf{E}_i$. Then,

$$\mathbb{E}_{\mathsf{Q}}[\tau_\alpha] \geqslant \frac{\log(1/\alpha)}{\max_{\lambda \in \triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1\right)\right]}.$$

Let $\lambda_{\mathsf{Q}} := \operatorname{argmax}_{\lambda \in \triangle_d} \mathbb{E}_{\mathsf{Q}}[\log(\lambda^\top \mathbf{E}_1)]$. Given the Q-a.s. upper bound

$$\limsup_{n \to \infty} \frac{1}{n} \log(W_n) \leqslant \mathbb{E}_{\mathsf{Q}} \left[\log \left(\lambda_{\mathsf{Q}}^\top \mathbf{E}_1\right)\right]$$

and the rejection time lower bound

$$\mathbb{E}_{\mathsf{Q}}[\tau_\alpha] \geqslant \frac{\log(1/\alpha)}{\mathbb{E}_{\mathsf{Q}} \left[\log \left(\lambda_{\mathsf{Q}}^\top \mathbf{E}_1\right)\right]},$$

it is desirable to seek out tests that behave as if we knew λ_{Q} (the "Kelly bet").

Recall, we are in a setting with a *composite* alternative $\mathcal{Q}$. We do have access to $\mathcal{Q}$ but we do not know Q nor λ_{Q}.

Are these "learnable", and under what conditions on Q?

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I do not know for sure, but this will not matter.

For the purposes of deriving an optimal test, it suffices to choose $(\lambda_n)_{n\in\mathbb{N}}$ with *sublinear portfolio regret*.

Definition: Portfolio regret.

Define the portfolio regret $\mathcal{R}_n$ of $(\lambda_n)_{n\in\mathbb{N}}$ to be

$$\mathcal{R}_n := \max_{\lambda \in \triangle_d} \sum_{i=1}^{n} \log\left(\lambda^\top \mathbf{E}_i\right) - \sum_{i=1}^{n} \log\left(\lambda_i^\top \mathbf{E}_i\right).$$

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Note, this is defined pathwise (irrespective of $\mathsf{Q}, \mathcal{Q}, \mathcal{P}$*, etc.). It is not obvious whether this is a desirable quantity to minimize.*

Theorem: Log-optimality via sublinear portfolio regret.

Suppose that $(\lambda_n)_{n\in\mathbb{N}}$ have sublinear portfolio regret. Then for $\mathsf{Q} \in \mathcal{Q}$,

$$\lim_{n\to\infty} \frac{1}{n} \log W_n = \max_{\lambda\in\triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1\right)\right] \quad \mathsf{Q}\text{-almost surely.}$$

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Moreover,

$$\lim_{\alpha\to 0^+} \frac{\mathbb{E}_{\mathsf{Q}}\left[\tau_\alpha\right]}{\log(1/\alpha)} = \frac{1}{\max_{\lambda\in\triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1\right)\right]}.$$

In particular, one could take λ_n according to Cover's universal portfolio algorithm:

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$$\lambda_n := \frac{\int_{\lambda \in \triangle_d} \lambda W_{n-1}(\lambda) \mathrm{d}F(\lambda)}{\int_{\lambda \in \triangle_d} W_{n-1}(\lambda) \mathrm{d}F(\lambda)},$$

in which case, the portfolio regret is *logarithmic*: $\mathcal{R}_n = O(\log n)$.

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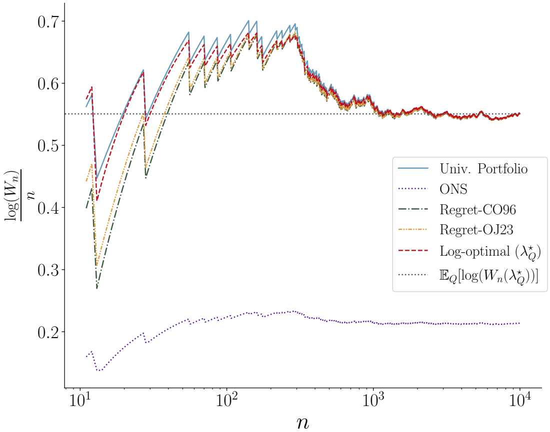
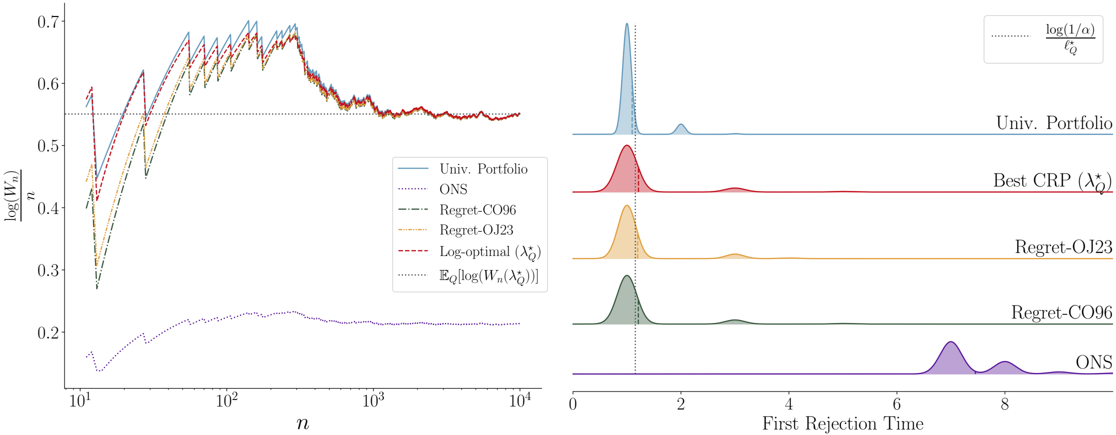
Suggested by Orabona & Jun [2023] for confidence sequences.



0.7
0.6
0.5
0.4
0.3
0.2
$\frac{\log(W_n)}{n}$
10^1
10^2
10^3
10^4
n
Univ. Portfolio
ONS
Regret-CO96
Regret-OJ23
Log-optimal ($\lambda_Q^\star$)
$\mathbb{E}_Q[\log(W_n(\lambda_Q^\star))]$



0.7
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Univ. Portfolio
Best CRP ($\lambda_Q^\star$)
Regret-OJ23
Regret-CO96
ONS
0
2
4
6
8
First Rejection Time

Recap so far

1. **Q:** "What is sequential hypothesis testing?"
   A: For all stopping times τ, $\mathsf{P}(\phi_\tau^{(\alpha)} \text{ rejects}) \leqslant \alpha$.

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Recap so far

1. **Q:** "What is sequential hypothesis testing?"
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2. **Q:** "How to derive valid sequential tests?"
   A: $W_n \mapsto \mathbb{1}\{W_n \geqslant 1/\alpha\}$ for a test supermartingale $(W_n)_{n\in\mathbb{N}}$.

3. **Q:** "How to derive a powerful supermartingale $(W_n)_{n\in\mathbb{N}}$?"
   A: Sublinear portfolio regret (e.g. Universal Portfolio algorithms).

Outline

1. What is sequential hypothesis testing (by betting)?
2. How are sequential hypothesis tests derived?
3. Defining and deriving optimal sequential tests.
4. **Multi-armed sequential hypothesis testing**

0mg

0.2mg

0.4mg

0.6mg

0.8mg

1mg

0mg 0.2mg 0.4mg 0.6mg 0.8mg 1mg

$$\mathcal{P} = \{\mathsf{P} : \textit{All} \text{ dosages are ineffective under } \mathsf{P}\}$$

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0mg 0.2mg 0.4mg 0.6mg 0.8mg 1mg

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(Also referred to as "Demonstration Experiments"; see Imbens–Masoero–Rakhlin–Richardson–Vijaykumar [2026].)

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How does this complicate type-I error control under $\mathcal{P}$ and/or log-optimality under $\mathcal{Q}$?

Let $\mathcal{A} = \{1, \ldots, K\}$ be the arm set.

Multi-armed sequential testing.

Start with $W_1 = \$1$.

For $n = 1, 2, \ldots$:

1. **Choose arm $A_n \in \mathcal{A}$.**
2. Choose portfolio $\lambda_n \in \triangle_d$.
3. Observe $\mathbf{E}_n(A_n) \sim \mathsf{P}(A_n)$.
4. $W_n = W_{n-1} {\lambda_n}^\top \mathbf{E}_n(A_n)$.

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EndFor

Now, $W_n = \prod_{i=1}^{n} {\lambda_i}^\top \mathbf{E}_i(A_i)$

Proposition Type-I error control under $\mathcal{P}$

No matter how λ_n and $A_n \in \mathcal{A}$ are chosen* ,

$$W_n = \prod_{i=1}^{n} \lambda_i{}^\top \mathbf{E}_i(A_i)$$

forms a test supermartingale and hence

$$\phi_n^{(\alpha)} := \mathbb{1}\{W_n \geqslant 1/\alpha\}$$

forms a sequential hypothesis test for the global null $\mathcal{P}$.

**formally, as long as they are measurable with respect to* $\mathbf{E}_1(A_1), \ldots, \mathbf{E}_{n-1}(A_{n-1})$

Takeaway: type-I error control in the multi-armed setting is obtained for free.

Optimality is a different story.

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Optimality is a different story.

Let us inspect the result first, and discuss how it is achieved later.

Theorem: Multi-armed log-optimality

Fix $\mathsf{Q} \in \mathcal{Q}$ inducing a distribution on all arms $\mathcal{A}$.

Choose $(\lambda_n(a))_{n \in \mathbb{N}}$ with sublinear portfolio regret for each $a \in \mathcal{A}$.

Choose $(A_n)_{n \in \mathbb{N}}$ according to a bespoke upper-confidence-bound-type allocation (more details later...)

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$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \log\left(\lambda_i(a)^\top \mathbf{E}_i(A_i)\right) = \max_{(a,\lambda)\in\mathcal{A}\times\triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1(a)\right)\right]$$

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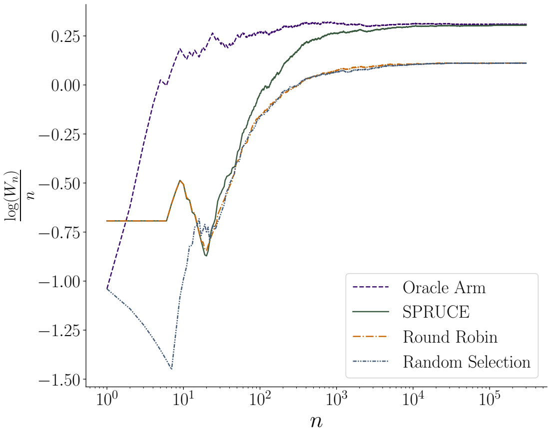
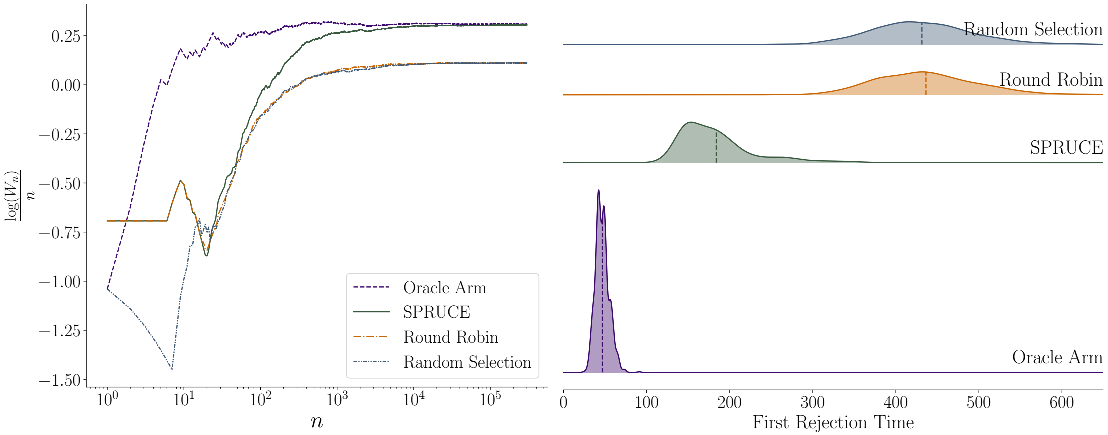
with Q-probability one. Furthermore,

$$\lim_{\alpha\to 0^+} \frac{\mathbb{E}_{\mathsf{Q}}[\tau_\alpha]}{\log(1/\alpha)} = \left(\max_{(a,\lambda)\in\mathcal{A}\times\triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1(a)\right)\right]\right)^{-1}$$



$\frac{\log(W_n)}{n}$
0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.25
−1.50
10^0
10^1
10^2
10^3
10^4
10^5
n
Oracle Arm
SPRUCE
Round Robin
Random Selection



0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.25
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$\frac{\log(W_n)}{n}$
10^0
10^1
10^2
10^3
10^4
10^5
n
Oracle Arm
SPRUCE
Round Robin
Random Selection
Random Selection
Round Robin
SPRUCE
Oracle Arm
0
100
200
300
400
500
600
First Rejection Time

What is the idea behind the algorithm and proof?

We are tasked with bounding the difference

$$\max_{(a,\lambda)\in\mathcal{A}\times\triangle_d} n\mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1(a)\right)\right] - \sum_{i=1}^{n} \log\left({\lambda_i}^\top \mathbf{E}_i(A_i)\right).$$

$$
\begin{aligned}
&\max_{(a,\lambda)\in\mathcal{A}\times\triangle_d} n\mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1(a)\right)\right] - \sum_{i=1}^n \log\left(\lambda_i{}^\top \mathbf{E}_i(A_i)\right) \\
= &\underbrace{\max_{(a,\lambda)\in\mathcal{A}\times\triangle_d} n\mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1(a)\right)\right] - \sum_{i=1}^n \mathbb{E}_{\mathsf{Q}}\left[\log(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}(A_i))\right]}_{\textbf{Allocation (multi-armed bandit) regret}} \\
+ &\underbrace{\sum_{i=1}^n \mathbb{E}_{\mathsf{Q}}\left[\log(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}_i(A_i))\right] - \sum_{i=1}^n \log\left(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}_i(A_i)\right)}_{\approx\, 0\ \mathsf{Q}\text{-a.s. by the strong law of large numbers.}} \\
+ &\underbrace{\sum_{i=1}^n \log\left(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}_i(A_i)\right) - \sum_{i=1}^n \log\left(\lambda_i{}^\top \mathbf{E}_i(A_i)\right)}_{\text{Arm-wise portfolio regret}}
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We are now focused on minimizing

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This is reminiscent of the type of regret considered in stochastic multi-armed bandits.

So, should we just use UCB for allocation?

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Not so fast...

In "vanilla" stochastic multi-armed bandits, we have

1. $X_1(a), \ldots, X_n(a) \overset{\text{iid}}{\sim} \mathsf{P}(a)$ for each arm $a \in \mathcal{A}$.
2. Assume something about all $\mathsf{P}(a)$ — bddness, sub-Gaussian, etc.

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But recall the allocation regret in our setting:

$$\max_{a\in\mathcal{A}} n\mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda_{\mathsf{Q}}^\top \mathbf{E}_1(a)\right)\right] - \sum_{i=1}^n \mathbb{E}_{\mathsf{Q}}\left[\log(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}(A_i))\right].$$

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So for us, $X_i = \log \left(\lambda_{\mathsf{Q}}(a)^{\top} \mathbf{E}_i(a) \right)$.

These are *neither* sub-Gaussian *nor* observed iid from any distribution.

Proposition: Confidence intervals for optimal growth rates

Suppose $\lambda_{\mathsf{Q}}^{\top}\mathbf{E}_1 \leqslant b$ for some $b > 1$. Define for $\alpha \in (0,1)$,

$$C_n(\alpha) := \max_{\lambda \in \triangle_d} \frac{1}{n}\sum_{i=1}^{n} \log(\lambda^{\top}\mathbf{E}_i) \pm \sqrt{\frac{8b\log(1/\alpha)}{n} + \frac{4\log(n+1)}{n}}$$

Then, with Q-probability $\geqslant 1-\alpha$,

$$\mathbb{E}_{\mathsf{Q}}\left[\log(\lambda_{\mathsf{Q}}^{\top}\mathbf{E}_1\right] \in C_n(\alpha)$$

We are now equipped with a way to derive a UCB algorithm.

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Define the upper confidence bound

$$\mathrm{UCB}_a(n) := \max_{\lambda \in \triangle_d} \frac{1}{N_a(n-1)} \sum_{i=1}^{n} \mathbb{1}\{A_i = a\} \log(\lambda^\top \mathbf{E}_i(A_i)) + \sqrt{\frac{8b \log(n+1)}{N_a(n-1)}} + \frac{4 \log(n+1)}{N_a(n-1)}.$$

We are now equipped with a way to derive a UCB algorithm.

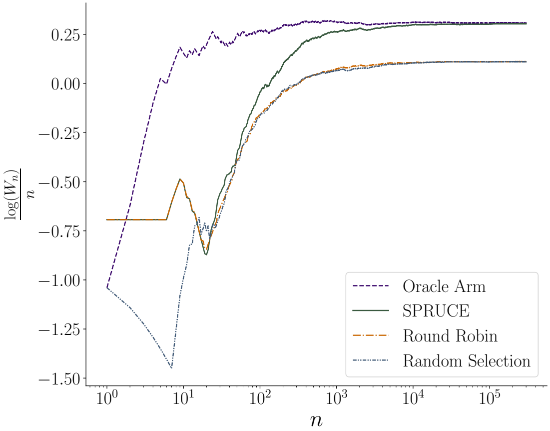
Let $N_a(n) := \sum_{i=1}^{n} \mathbb{1}\{A_i = a\}$ be the number of pulls for arm a.

Define the upper confidence bound

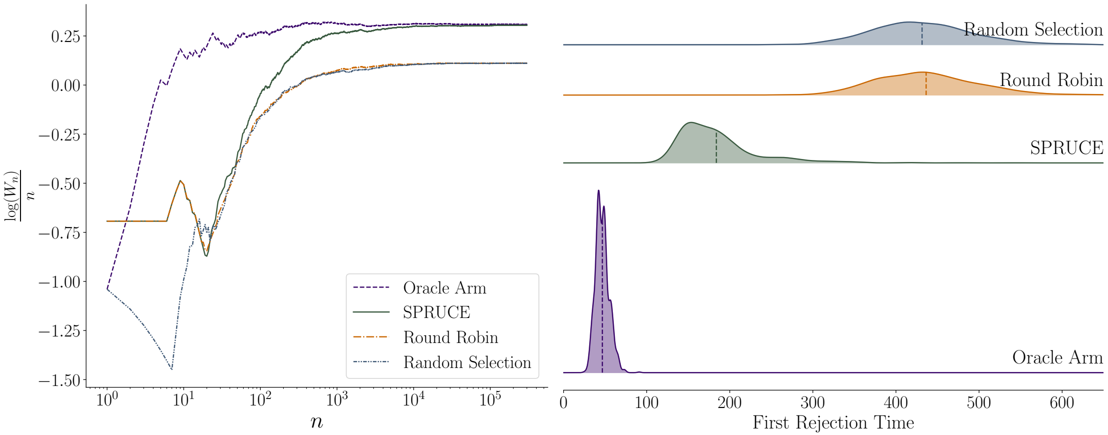
$$\mathrm{UCB}_a(n) := \max_{\lambda \in \triangle_d} \frac{1}{N_a(n-1)} \sum_{i=1}^{n} \mathbb{1}\{A_i = a\} \log(\lambda^\top \mathbf{E}_i(A_i)) \\ + \sqrt{\frac{8b \log(n+1)}{N_a(n-1)}} + \frac{4 \log(n+1)}{N_a(n-1)}.$$

This turns out to have all the "nice" properties needed for UCB-style analyses. In particular, it suffices for

$$\text{Allocation regret}_n = O\left(\log(n)\right).$$



0.25
0.00
−0.25
−0.50
−0.75
−1.00
−1.25
−1.50
$\frac{\log(W_n)}{n}$
10^0
10^1
10^2
10^3
10^4
10^5
n
Oracle Arm
SPRUCE
Round Robin
Random Selection



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10^2
10^3
10^4
10^5
n
Oracle Arm
SPRUCE
Round Robin
Random Selection
Random Selection
Round Robin
SPRUCE
Oracle Arm
0
100
200
300
400
500
600
First Rejection Time

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   A: Sublinear portfolio regret (e.g. Universal Portfolio algorithms).

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   A: Sublinear portfolio regret (e.g. Universal Portfolio algorithms).

4. **Q:** "How to do the same when multiple arms are present?"
   A: Bespoke upper-confidence-bound algorithms.

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