Log-Optimality & Multi-Armed Sequential Hypothesis Testing

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Outline

1. What is sequential hypothesis testing (by betting)?
2. How are sequential hypothesis tests derived?
3. Defining and deriving optimal sequential tests.
4. Multi-armed sequential hypothesis testing

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A motivating example to keep in mind: **treatment effect testing**.

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$$H_0 : \text{trt effect} = 0$$
$$H_1 : \text{trt effect} \neq 0$$

$$\alpha := 0.01$$

Step 2: Recruit n patients and randomize (**trt** or **ctrl**).

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Compute a p-value: $p_n = 0.023$

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No, this is "p-hacking".

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Sequential testing provides one solution to these scenarios.

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(Modern breakthroughs in recent years; textbook by Ramdas & Wang [2025])

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$$\textit{(e.g. } \mathcal{P} = \{P : \textit{trt effect} = 0\} \textit{ versus } \mathcal{Q} = \{P : \textit{trt effect} > 0\}\textit{)}$$

We are tasked with finding a test $\phi_n^{(\alpha)} \equiv \phi^{(\alpha)}(X_1, \ldots, X_n)$ that outputs 1 (rejects $\mathcal{P}$ in favour of $\mathcal{Q}$) with small probability under $\mathcal{P}$.

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Fixed-n test: $\forall n \in \mathbb{N},\ \sup_{\mathsf{P} \in \mathcal{P}} \mathsf{P}\left(\phi_n^{(\alpha)} \text{ rejects}\right) \leqslant \alpha.$

Sequential test: $\sup_{\mathsf{P} \in \mathcal{P}} \mathsf{P}\left(\exists n \in \mathbb{N} : \phi_n^{(\alpha)} \text{ rejects}\right) \leqslant \alpha.$

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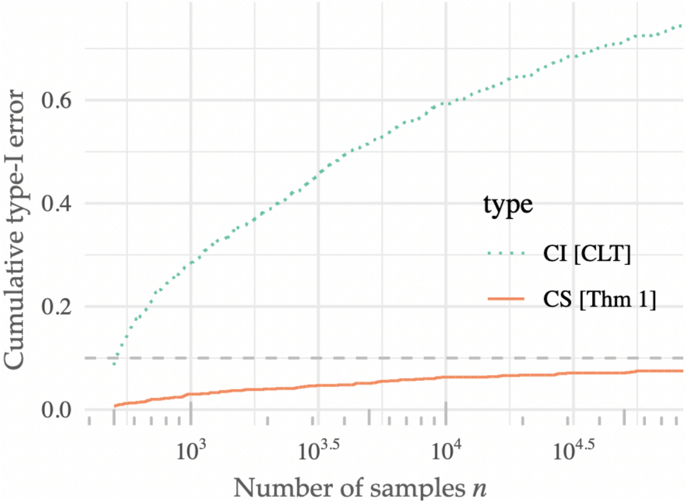
$$\iff \sup_{\mathsf{P} \in \mathcal{P}} \mathsf{P}\left(\phi_\tau^{(\alpha)} \text{ rejects}\right) \leqslant \alpha \quad \forall \tau.$$



$$\forall n,\ \mathsf{P}(\phi_n^{(\alpha)} = 1) \leqslant \alpha$$

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2. Set the test as $\phi_n^{(\alpha)} := \mathbb{1}\{W_n \geqslant 1/\alpha\}$.

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When $x = 1/\alpha$ and $\mathbb{E}_{\mathsf{P}}[W_1] \leqslant 1$ for each $\mathsf{P} \in \mathcal{P}$, we get

$$\forall \alpha \in (0,1), \quad \sup_{\mathsf{P}\in\mathcal{P}} \mathsf{P}\left(\exists n \in \mathbb{N} : \phi_n^{(\alpha)} = 1\right) \leqslant \alpha.$$

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However, lots of progress has been made in recent years.

Example: Testing the mean of a bounded random variables.

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Let $X_1, X_2, \dots \in [0,1]$. The null is $\mathcal{P} := \{\mathsf{P} : \mathbb{E}_{\mathsf{P}}[X_1] = 1/2\}$. Then,

$$W_n := \prod_{i=1}^{n} (1 + \gamma_i \cdot (2X_i - 1))$$

forms a test martingale for any $[0,1]$-valued *predictable* $(\gamma_n)_{n\in\mathbb{N}}$.

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Therefore, $\phi_n^{(\alpha)} := \mathbb{1}\{W_n \geqslant 1/\alpha\}$ yields a sequential test for $\mathcal{P}$.

It is **not** obvious how to choose γ_i... more on that later.

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&\mathbb{E}_{\mathsf{P}}\left[\prod_{i=1}^n (1 + \gamma_i(2X_i - 1)) \mid X_1, \dots, X_{n-1}\right] \\
&= W_{n-1}\mathbb{E}_{\mathsf{P}}\left[1 - \gamma_n(2X_i - 1) \mid X_1, \dots, X_{n-1}\right] \\
&= W_{n-1}\left(1 - \gamma_n(2\mathbb{E}_{\mathsf{P}}[X_n] - 1)\right) \\
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Therefore, W_n forms a test martingale. □

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Definition: e-value. (See Shafer & Vovk, Grünwald et al., Vovk & Wang)

A nonnegative random variable $E \geqslant 0$ is said to be a $\mathcal{P}$-e-value if

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One can re-cast several testing problems *(bounded means, two-sample, independence, equality of bounded tuples, testing randomness online, etc.)* from the literature with convex combinations of carefully chosen e-values.

Throughout, consider the process $(W_n)_{n\in\mathbb{N}}$ given by (predictable) convex combination of $d+1$ e-values:

$$W_n := \prod_{i=1}^{n} \lambda_i{}^\top \mathbf{E}_i,$$

$$\text{where} \quad \mathbf{E}_i = (E_0, \ldots, E_d) \quad \text{is a tuple of } \mathcal{P}\text{-e-values,}$$

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Some other special cases found in the literature

One-sided bounded mean testing: Set $\mathbf{E}_i = (1, X_i/\mu_0)$.
Two-sided bounded mean testing: Set $\mathbf{E}_i = ((1 - X_i)/(1 - \mu_0), X_i/\mu_0)$.
Two-sample testing: Set $\mathbf{E}_i = (1, g^\star(X_i) - g^\star(Y_i))$ for a witness f'n $g^\star$.

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Indeed, $\prod_{i=1}^{n} {\lambda_i}^\top \mathbf{E}_i$ yields a valid sequential test under $\mathcal{P}$ for *any* choice of $(\lambda_n)_{n\in\mathbb{N}}$.

Are there "better" choices than others under $\mathcal{Q}$?

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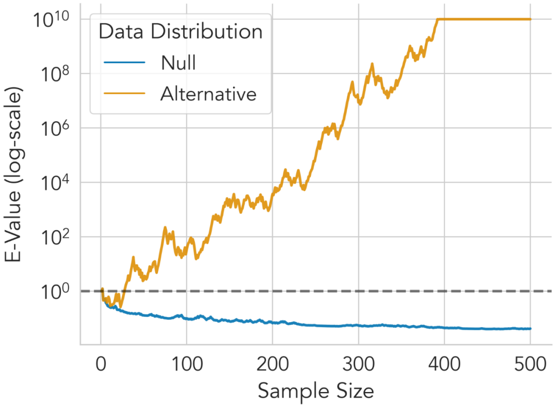
We show that these are optimized via the exact same criterion.

(i) Growth-rate-optimality.

(Kelly ['56], Long Jr. ['90], Grünwald et al. [2024], Larsson et al. [2024])



*An e-process is expected to be small under the **null**;*
*we want it to grow large under the **alternative**.*

Image credit: YJ Choe.

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Observe by the strong law of large numbers:

$$\begin{aligned} W_n &= \exp\left\{ n \cdot \frac{1}{n} \sum_{i=1}^{n} \log\left({\lambda_i}^\top \mathbf{E}_i \right) \right\} \\ &\approx \exp\left\{ n \cdot \mathbb{E}_{\mathsf{Q}} \left[\log\left({\lambda_i}^\top \mathbf{E}_i \right) \right] + o(n) \right\} \end{aligned}$$

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So, if data comes from $\mathsf{Q} \in \mathcal{Q}$, it is reasonable to want to choose λ_i so as to maximize:

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This is an instantiation of the "Kelly criterion" (J.L. Kelly Jr., [1956]).

Proposition: Upper bound on the asymptotic growth rate

Suppose $\mathbf{E}_1, \mathbf{E}_2, \cdots \sim \mathsf{Q} \in \mathcal{Q}$. For any predictable sequence $(\lambda_n)_{n\in\mathbb{N}}$, we have that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log\left({\lambda_i}^\top \mathbf{E}_i\right) \leqslant \max_{\lambda \in \triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log(\lambda^\top \mathbf{E}_1)\right]$$

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One can also motivate the Kelly criterion from a more “purely statistical” perspective.

(ii) Measuring optimality through expected rejection times

(Wald 1945, Breiman 1961, Kaufmann, Agrawal, Koolen, others from the bandit literature)

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Proposition: Lower bound on the expected stopping time

For any choice of $(\lambda_n)_{n\in\mathbb{N}}$, let τ_α be the resulting first hitting time of $W_n := \prod_{i=1}^n \lambda_i^\top \mathbf{E}_i$. Then,

$$\mathbb{E}_{\mathsf{Q}}[\tau_\alpha] \geqslant \frac{\log(1/\alpha)}{\max_{\lambda\in\triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1\right)\right]}.$$

Let $\lambda_{\mathsf{Q}} := \operatorname{argmax}_{\lambda \in \triangle_d} \mathbb{E}_{\mathsf{Q}}[\log(\lambda^\top \mathbf{E}_1)]$. Given the Q-a.s. upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \log(W_n) \leqslant \mathbb{E}_{\mathsf{Q}} \left[\log \left(\lambda_{\mathsf{Q}}^\top \mathbf{E}_1\right)\right]$$

and the rejection time lower bound

$$\mathbb{E}_{\mathsf{Q}}[\tau_\alpha] \geqslant \frac{\log(1/\alpha)}{\mathbb{E}_{\mathsf{Q}} \left[\log \left(\lambda_{\mathsf{Q}}^\top \mathbf{E}_1\right)\right]},$$

it is desirable to seek out tests that behave as if we knew λ_{Q} (the "Kelly bet").

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For the purposes of deriving an optimal test, it suffices to choose $(\lambda_n)_{n \in \mathbb{N}}$ with *sublinear portfolio regret*.

Definition: Portfolio regret.

Define the portfolio regret $\mathcal{R}_n$ of $(\lambda_n)_{n\in\mathbb{N}}$ to be

$$\mathcal{R}_n := \max_{\lambda\in\triangle_d} \sum_{i=1}^{n} \log\left(\lambda^\top \mathbf{E}_i\right) - \sum_{i=1}^{n} \log\left(\lambda_i{}^\top \mathbf{E}_i\right).$$

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This is *precisely* the notion of regret considered by Thomas Cover and Erik Ordentlich in their work on on universal portfolios circa 1990s.

Definition: Portfolio regret.

Define the portfolio regret $\mathcal{R}_n$ of $(\lambda_n)_{n\in\mathbb{N}}$ to be

$$\mathcal{R}_n := \max_{\lambda\in\triangle_d} \sum_{i=1}^{n} \log\left(\lambda^\top \mathbf{E}_i\right) - \sum_{i=1}^{n} \log\left(\lambda_i{}^\top \mathbf{E}_i\right).$$

This is *precisely* the notion of regret considered by Thomas Cover and Erik Ordentlich in their work on on universal portfolios circa 1990s.

Note, this is defined pathwise (irrespective of $\mathsf{Q}, \mathcal{Q}, \mathcal{P}$*, etc.). It is not obvious whether this is a desirable quantity to minimize.*

Theorem: Log-optimality via sublinear portfolio regret.

Suppose that $(\lambda_n)_{n\in\mathbb{N}}$ have sublinear portfolio regret. Then for $\mathsf{Q} \in \mathcal{Q}$,

$$\lim_{n\to\infty} \frac{1}{n} \log W_n = \max_{\lambda\in\triangle_d} \mathbb{E}_{\mathsf{Q}} \left[\log\left(\lambda^\top \mathbf{E}_1\right)\right] \quad \mathsf{Q}\text{-almost surely.}$$

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Moreover,

$$\lim_{\alpha\to 0^+} \frac{\mathbb{E}_{\mathsf{Q}}\left[\tau_\alpha\right]}{\log(1/\alpha)} = \frac{1}{\max_{\lambda\in\triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1\right)\right]}.$$

In particular, one could take λ_n according to Cover's universal portfolio algorithm:

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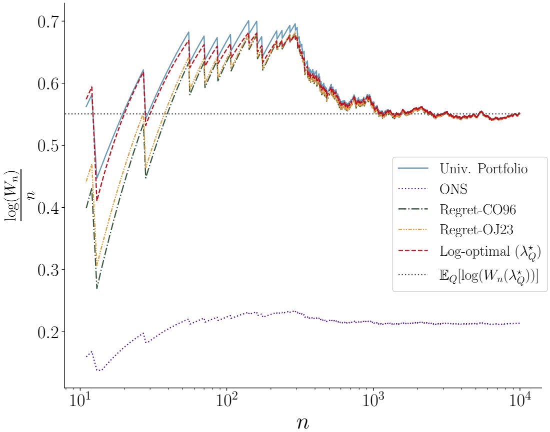
$$\lambda_n := \frac{\int_{\lambda \in \triangle_d} \lambda W_{n-1}(\lambda) \mathrm{d}F(\lambda)}{\int_{\lambda \in \triangle_d} W_{n-1}(\lambda) \mathrm{d}F(\lambda)},$$

in which case, the portfolio regret is *logarithmic*: $\mathcal{R}_n = O(\log n)$.



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ONS
Regret-CO96
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Log-optimal ($\lambda_Q^\star$)
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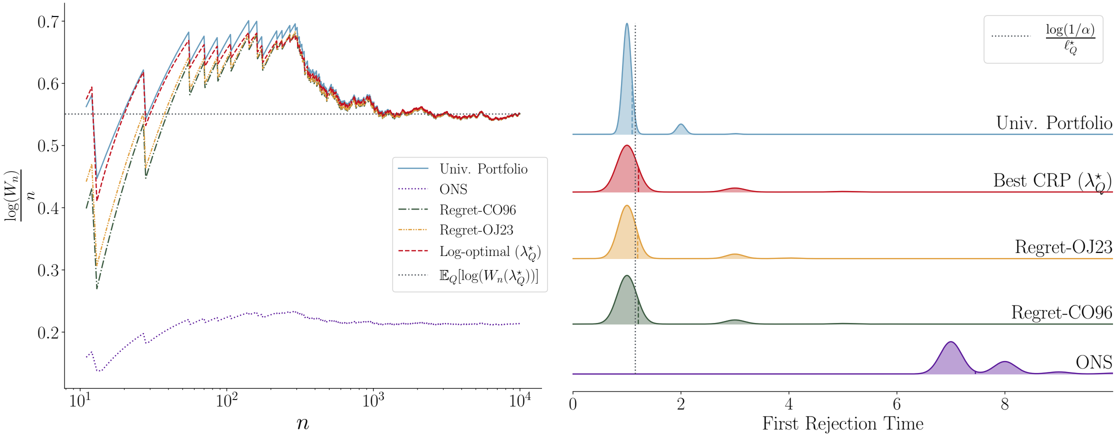

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First Rejection Time

Recap so far

1. **Q:** "What is sequential hypothesis testing?"
   A: For all stopping times τ, $\mathsf{P}(\phi_\tau^{(\alpha)} \text{ rejects}) \leqslant \alpha$.

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Recap so far

1. **Q:** "What is sequential hypothesis testing?"
   A: For all stopping times τ, $\mathsf{P}(\phi_\tau^{(\alpha)} \text{ rejects}) \leqslant \alpha$.

2. **Q:** "How to derive valid sequential tests?"
   A: $W_n \mapsto \mathbb{1}\{W_n \geqslant 1/\alpha\}$ for a test supermartingale $(W_n)_{n\in\mathbb{N}}$.

3. **Q:** "How to derive a powerful supermartingale $(W_n)_{n\in\mathbb{N}}$?"
   A: Sublinear portfolio regret (e.g. Universal Portfolio algorithms).

Outline

1. What is sequential hypothesis testing (by betting)?
2. How are sequential hypothesis tests derived?
3. Defining and deriving optimal sequential tests.
4. **Multi-armed sequential hypothesis testing**

0mg

0.2mg

0.4mg

0.6mg

0.8mg

1mg

0mg 0.2mg 0.4mg 0.6mg 0.8mg 1mg

$\mathcal{P} = \{\mathsf{P} : \textit{All} \text{ dosages are ineffective under } \mathsf{P}\}$

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How does this complicate type-I error control under $\mathcal{P}$ and/or log-optimality under $\mathcal{Q}$?

Let $\mathcal{A} = \{1, \ldots, K\}$ be the arm set.

Multi-armed sequential testing.

Start with $W_1 = \$1$.

For $n = 1, 2, \ldots$:

1. **Choose arm $A_n \in \mathcal{A}$.**
2. Choose portfolio $\lambda_n \in \triangle_d$.
3. Observe $\mathbf{E}_n(A_n) \sim \mathsf{P}(A_n)$.
4. $W_n = W_{n-1}\lambda_n{}^\top \mathbf{E}_n(A_n)$.

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EndFor

Now, $W_n = \prod_{i=1}^n \lambda_i^\top \mathbf{E}_i(A_i)$

Proposition Type-I error control under $\mathcal{P}$

No matter how λ_n and $A_n \in \mathcal{A}$ are chosen* ,

$$W_n = \prod_{i=1}^{n} {\lambda_i}^\top \mathbf{E}_i(A_i)$$

forms a test supermartingale and hence

$$\phi_n^{(\alpha)} := \mathbb{1}\{W_n \geqslant 1/\alpha\}$$

forms a sequential hypothesis test for the global null $\mathcal{P}$.

* *formally, as long as they are measurable with respect to* $\mathbf{E}_1(A_1), \ldots, \mathbf{E}_{n-1}(A_{n-1})$

Takeaway: type-I error control in the multi-armed setting is obtained for free.

Optimality is a different story.

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Optimality is a different story.

Let us inspect the result first, and discuss how it is achieved later.

Theorem: Multi-armed log-optimality

Fix $\mathsf{Q} \in \mathcal{Q}$ inducing a distribution on all arms $\mathcal{A}$.

Choose $(\lambda_n(a))_{n\in\mathbb{N}}$ with sublinear portfolio regret for each $a \in \mathcal{A}$.

Choose $(A_n)_{n\in\mathbb{N}}$ according to a bespoke upper-confidence-bound-type allocation (more details later...)

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with Q-probability one.

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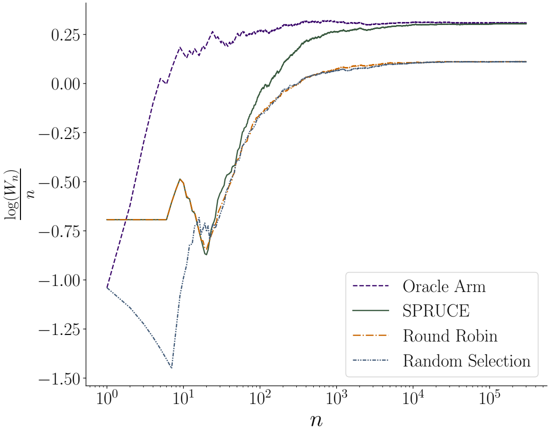
with Q-probability one. Furthermore,

$$\lim_{\alpha\to 0^+} \frac{\mathbb{E}_{\mathsf{Q}}[\tau_\alpha]}{\log(1/\alpha)} = \left(\max_{(a,\lambda)\in\mathcal{A}\times\triangle_d} \mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1(a)\right)\right]\right)^{-1}$$



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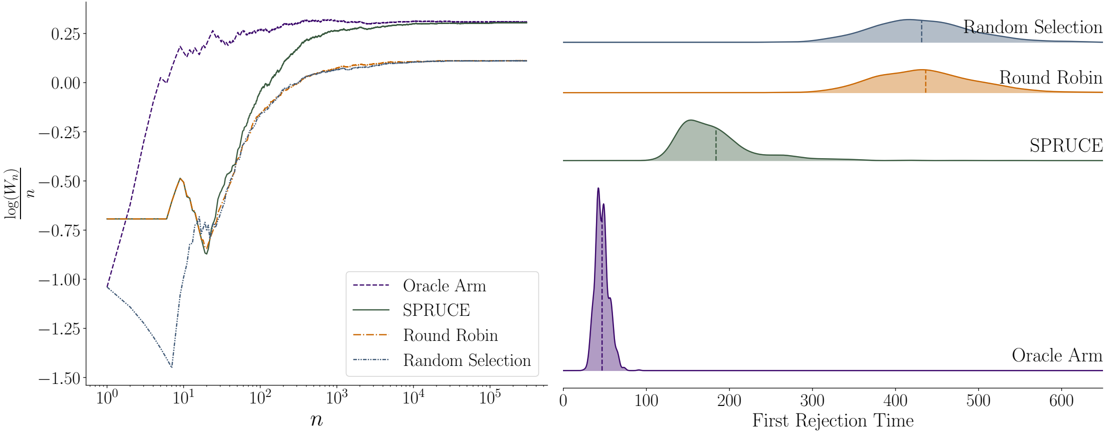

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What is the idea behind the algorithm and proof?

We are tasked with bounding the difference

$$\max_{(a,\lambda)\in\mathcal{A}\times\triangle_d} n\mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1(a)\right)\right] - \sum_{i=1}^{n} \log\left(\lambda_i{}^\top \mathbf{E}_i(A_i)\right).$$

$$
\begin{aligned}
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&= \underbrace{\max_{(a,\lambda)\in\mathcal{A}\times\triangle_d} n\mathbb{E}_{\mathsf{Q}}\left[\log\left(\lambda^\top \mathbf{E}_1(a)\right)\right] - \sum_{i=1}^{n} \mathbb{E}_{\mathsf{Q}}\left[\log(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}(A_i))\right]}_{\textbf{Allocation (multi-armed bandit) regret}} \\
&+ \underbrace{\sum_{i=1}^{n} \mathbb{E}_{\mathsf{Q}}\left[\log(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}_i(A_i))\right] - \sum_{i=1}^{n} \log\left(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}_i(A_i)\right)}_{\approx\, 0\ \mathsf{Q}\text{-a.s. by the strong law of large numbers.}} \\
&+ \underbrace{\sum_{i=1}^{n} \log\left(\lambda_{\mathsf{Q}}(A_i)^\top \mathbf{E}_i(A_i)\right) - \sum_{i=1}^{n} \log\left(\lambda_i{}^\top \mathbf{E}_i(A_i)\right)}_{\text{Arm-wise portfolio regret}}
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We are now focused on minimizing

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This is *almost exactly* the type of regret considered in stochastic multi-armed bandits, which can typically be $O(\log n)$ when using upper-confidence-bound-type algorithms.

A one-slide crash course on upper-confidence-bound algorithms and the "principle of optimism" (Lai & Robbins, 1985).

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Illustration (credits: Adrien Prevost, INRIA)

</detour>

“Nice” confidence intervals $\implies$ allocation regret minimization.

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1. Assume sub-Gaussianity, and
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But recall the allocation regret:

$$\max_{a \in \mathcal{A}} n \mathbb{E}_{\mathsf{Q}} \left[\log \left(\lambda_{\mathsf{Q}}^{\top} \mathbf{E}_1(a) \right) \right] - \sum_{i=1}^{n} \mathbb{E}_{\mathsf{Q}} \left[\log(\lambda_{\mathsf{Q}}(A_i)^{\top} \mathbf{E}(A_i)) \right].$$

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These are *neither* sub-Gaussian *nor* observed iid from any distribution.

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$$\mathrm{UCB}_a(n) := \max_{\lambda \in \triangle_d} \frac{1}{N_a(n-1)} \sum_{i=1}^{n} \mathbb{1}\{A_i = a\} \log(\lambda^\top \mathbf{E}_i(A_i)) + \sqrt{\frac{8 \log(n+1)}{N_a(n-1)}} + \frac{4 \log(n+1)}{N_a(n-1)}.$$

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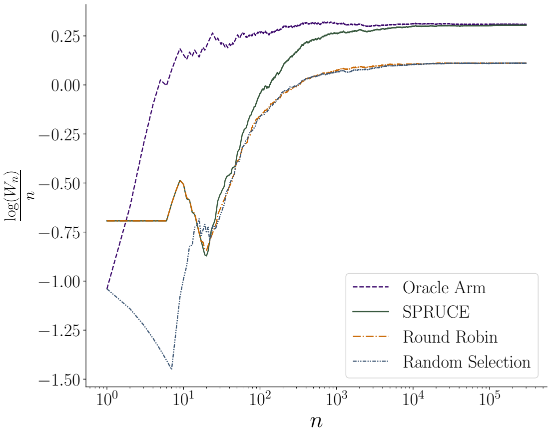
This turns out to have all the “nice” properties needed for UCB-style analyses. In particular, it suffices for

$$\text{Allocation regret}_n = O\left(\log(n)\right).$$



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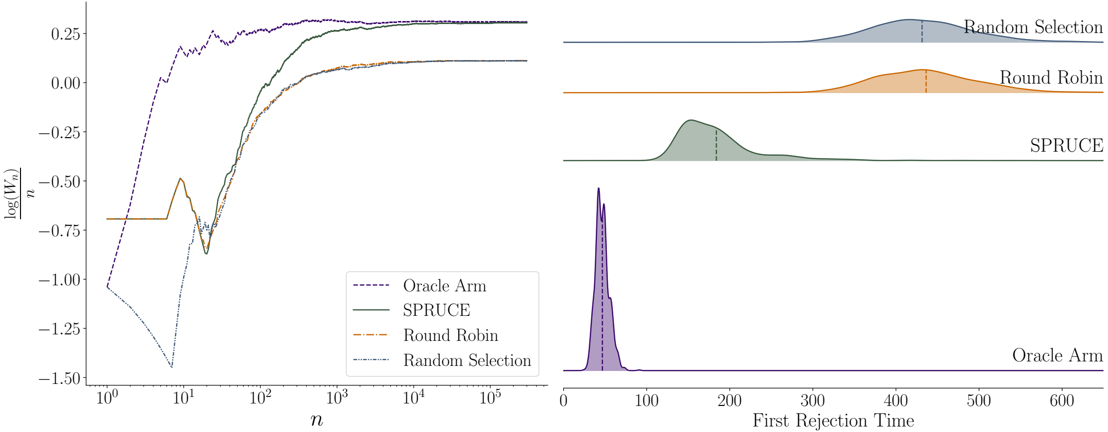

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3. **Q:** "How to derive a powerful supermartingale $(W_n)_{n\in\mathbb{N}}$?"
   A: Sublinear portfolio regret (e.g. Universal Portfolio algorithms).

Summary

1. **Q:** "What is sequential hypothesis testing?"
   A: For all stopping times τ, $\mathsf{P}(\phi_\tau^{(\alpha)} \text{ rejects}) \leqslant \alpha$.

2. **Q:** "How to derive valid sequential tests?"
   A: $W_n \mapsto \mathbb{1}\{W_n \geqslant 1/\alpha\}$ for a test supermartingale $(W_n)_{n\in\mathbb{N}}$.

3. **Q:** "How to derive a powerful supermartingale $(W_n)_{n\in\mathbb{N}}$?"
   A: Sublinear portfolio regret (e.g. Universal Portfolio algorithms).

4. **Q:** "How to do the same when multiple arms are present?"
   A: Bespoke upper-confidence-bound algorithms.

Thank you

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