Asymptotic anytime-valid statistical inference

Ian Waudby-Smith

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International Workshop on Statistical Methodologies 2026

A brief crash course on (nonasymptotic) anytime-valid inference

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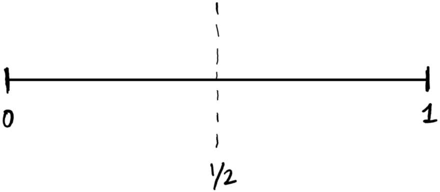

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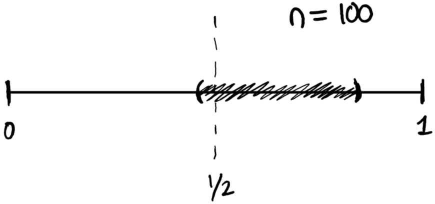

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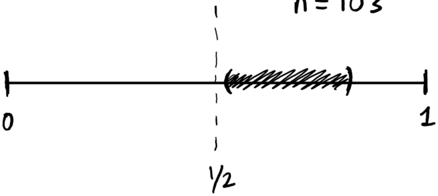

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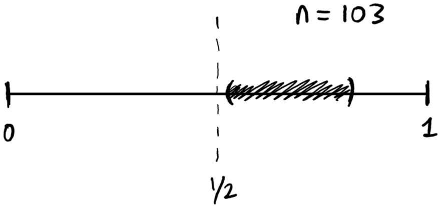
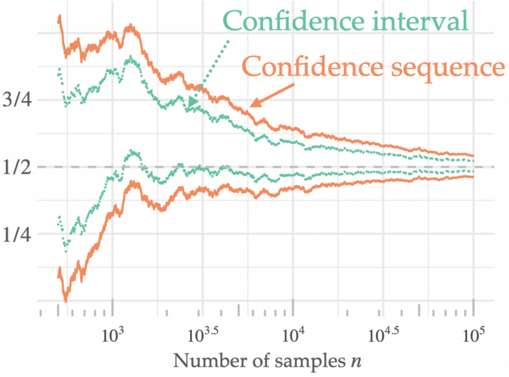


This is tempting, but **inflates type-I error**.

A solution: **Confidence sequences**

(Herbert Robbins & coauthors, 1960s-70s)

Estimation view



CI: $\forall n \in \mathbb{N},\ P(\theta \in \dot{C}_n) \geqslant 1 - \alpha.$

CS: $P(\forall n \in \mathbb{N},\ \theta \in \bar{C}_n) \geqslant 1 - \alpha.$

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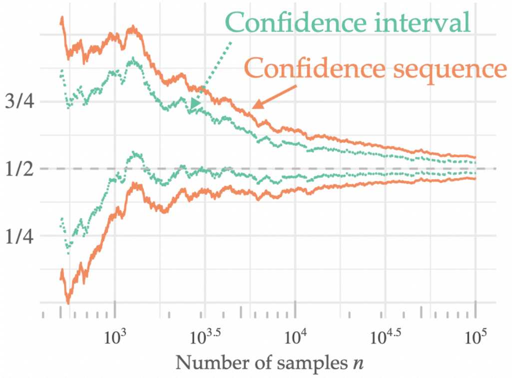

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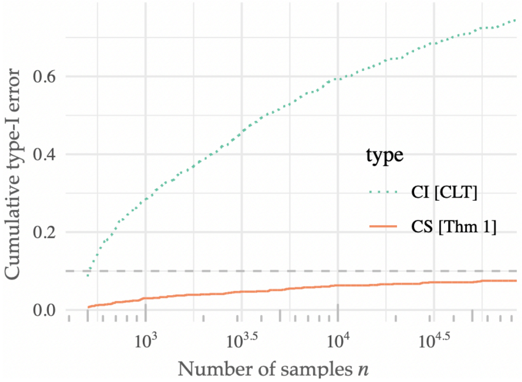

CI: $\forall n \in \mathbb{N},\ P(\theta \notin \dot{C}_n) \leqslant \alpha.$

CS: $P(\exists n \in \mathbb{N} : \theta \notin \bar{C}_n) \leqslant \alpha.$

Confidence sequences, sequential tests, etc. draw on a rich literature:

(i) Ville, Wald ('30–'40s)

(ii) Robbins, Siegmund, Darling, and Lai ('60s–'80s)

(iii) Vovk, Shafer, Johari, Howard, Kaufmann, Ramdas, Wang, Grünwald, de Heide, Koolen, Orabona, Jun, Duchi, Shekhar, **Agrawal**, **Ram** and **Wang** (2010s-now).

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The vast majority of this work has been *nonasymptotic*:

$$\mathbb{P}(\exists n \in \{1, 2, \dots\} : \mathbf{error}_n) \leqslant \alpha.$$

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Non-asymptotic inference is provably impossible	**Central limit theorem**	This talk.

Asymptotic anytime-valid inference

Robbins & Siegmund (1970)
W-S et al., (2024a/2024b)
Bibaut et al. (2022)

Time-uniform central limit theory
and asymptotic confidence sequences

Ian Waudby-Smith[1], David Arbour[2], Ritwik Sinha[2], Edward H. Kennedy[1], and Aaditya Ramdas[1]

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`edward@stat.cmu.edu`, `aramdas@cmu.edu`

Annals of Statistics, 2024

We will look at *doubly indexed* intervals for each $k \geqslant m$:

$$(C_k^{(1)})_{k=1}^{\infty}, (C_k^{(2)})_{k=2}^{\infty}, (C_k^{(3)})_{k=3}^{\infty}, \dots$$

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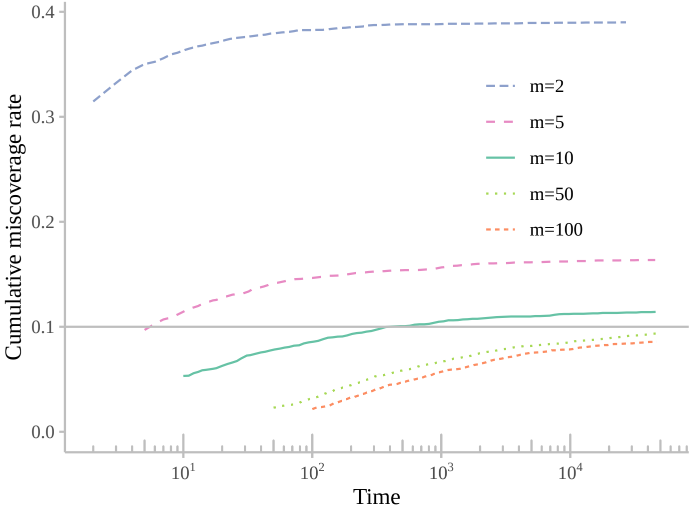
Definition. (W-S et al., AoS 2024a)

We say $C_k^{(m)}$ has $\underline{\text{asymptotic anytime } (1-\alpha)\text{-coverage}}$ for μ if

$$\forall P \in \mathcal{P}, \quad \limsup_{m \to \infty} P\left(\exists k \geqslant m : \mu \notin C_k^{(m)} \right) \leqslant \alpha.$$

Asymptotic anytime-valid inference in a picture



Example: Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. on $(\Omega, \mathcal{F}, P)$ with mean μ_P and variance σ_P^2.

Define $\widehat{\mu}_n := \frac{1}{n}\sum_{i=1}^n X_i$ and $\widehat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \widehat{\mu}_n^2$.

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If $\mathbb{E}_P|X|^3 < \infty$, then $(C_k^{(m)})_{k=m}^{\infty}$ has asymptotic coverage for μ_P where

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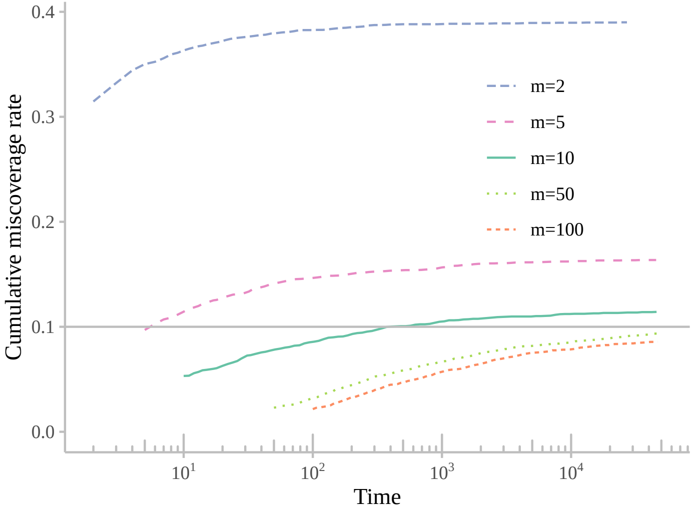
That is,

$$\forall P \in \mathcal{P}, \quad \lim_{m\to\infty} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)}\right) = \alpha.$$

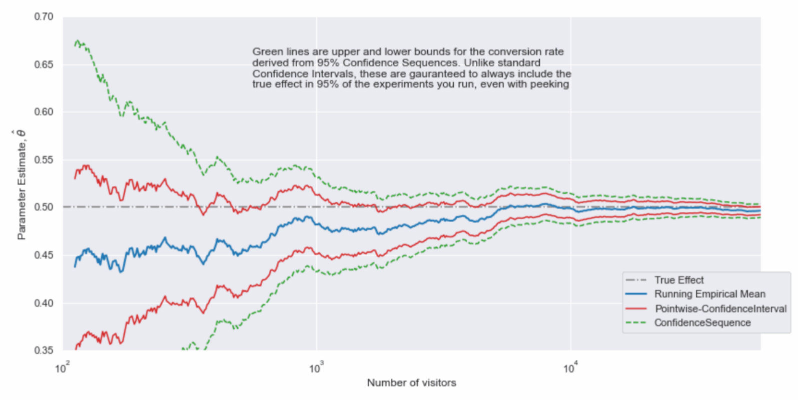
Asymptotic anytime-valid inference in a picture



Adobe's Statistical Methodology: Any Time Valid Confidence Sequences

A **Confidence Sequence** is a sequential analog of a **Confidence Interval**, e.g. if you repeat your experiments one hundred times, and calculate an estimate of the mean metric and its associated 95%-Confidence Sequence for every new user that enters the experiment. A 95% Confidence Sequence will include the true value of the metric in 95 out of the 100 experiments that you ran. A 95% Confidence Interval could only be calculated once per experiment in order to give the same 95% coverage guarantee; not with every single new user. Confidence Sequences therefore allow you to continuously monitor experiments, without increasing False Positive error rates.

The difference between confidence sequences and confidence intervals for a single experiment is shown in the animation below:



Green lines are upper and lower bounds for the conversion rate derived from 95% Confidence Sequences. Unlike standard Confidence Intervals, these are gauranteed to always include the true effect in 95% of the experiments you run, even with peeking

GrowthBook's implementation

There are many approaches to sequential testing, several of which are well explained and compared in this Spotify blogpost.

For GrowthBook, we selected a method that would work for the wide variety of experimenters that we serve, while also providing experimenters with a way to tune the approach for their setting. To that end, we implement Asymptotic Confidence Sequences introduced by Waudby-Smith et al. (2023); these are very similar to the Generalized Anytime Valid Inference confidence sequences described by Spotify in the above post and introduced by Howard et al. (2022), although the Waudby-Smith et al. approach more transparently applies to our setting.

Specifically, GrowthBook's confidence sequences, which take the place of confidence intervals, become:

$$\left(\hat{\mu} \pm \hat{\sigma} * \sqrt{N} * \sqrt{\frac{2(N\rho^2+1)}{N^2\rho^2} \log\left(\frac{\sqrt{N\rho^2+1}}{\alpha}\right)}\right)$$

$$\rho = \sqrt{\frac{-2\log(\alpha)+\log(-2\log(\alpha)+1)}{N^*}}$$

In the above, $\hat{\mu}$ is our estimated uplift, $\hat{\sigma}$ is the estimated standard error of the uplift, α is our significance level (defaults to 0.05), and N is the sum of the two variation's sample sizes.

However, notice that the guarantee

$$\forall P \in \mathcal{P}, \quad \lim_{m \to \infty} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)}\right) = \alpha$$

is *distribution-pointwise*.

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Many modern asymptotic methods aim to be *distribution-uniform*, in the sense of

$$\lim_{m \to \infty} \sup_{P \in \mathcal{P}} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)}\right) = \alpha.$$

How were we able to show that

$$C_k^{(m)} := \left[\widehat{\mu}_k \pm \widehat{\sigma}_k \sqrt{\frac{\Psi^{-1}(1 - \alpha) + \log(k/m)}{k}} \right]?$$

satisfies the $\mathcal{P}$-uniform coverage guarantee?

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Consider the function

$$\Psi(x) := 1 - 2\left[1 - \Phi(\sqrt{x}) + \sqrt{x}\phi(\sqrt{x})\right];\ x \geqslant 0,$$

where Φ is the Gaussian CDF and ϕ is the Gaussian density.

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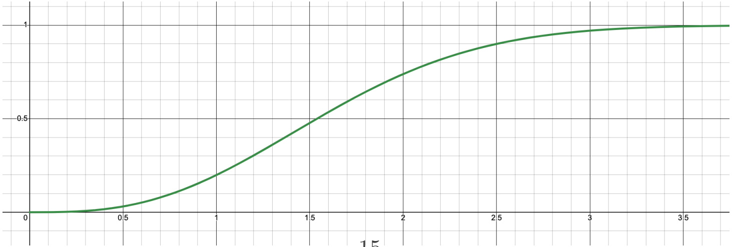

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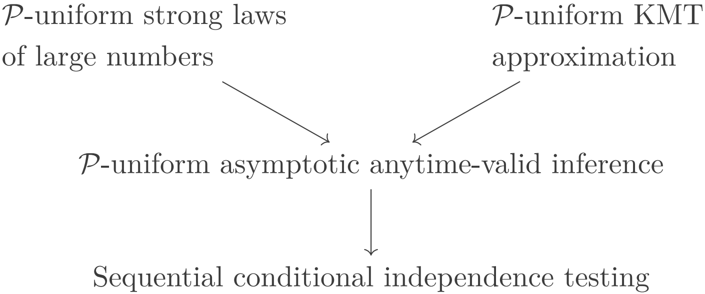
Then $\mathbb{P}(\zeta \leqslant x) = \Psi(x)$; $x \geqslant 0$.
(see Robbins & Siegmund (1970) or W-S et al., 2024a).

Some related lines of work



$\mathcal{P}$-uniform strong laws of large numbers

$\mathcal{P}$-uniform KMT approximation

$\mathcal{P}$-uniform asymptotic anytime-valid inference

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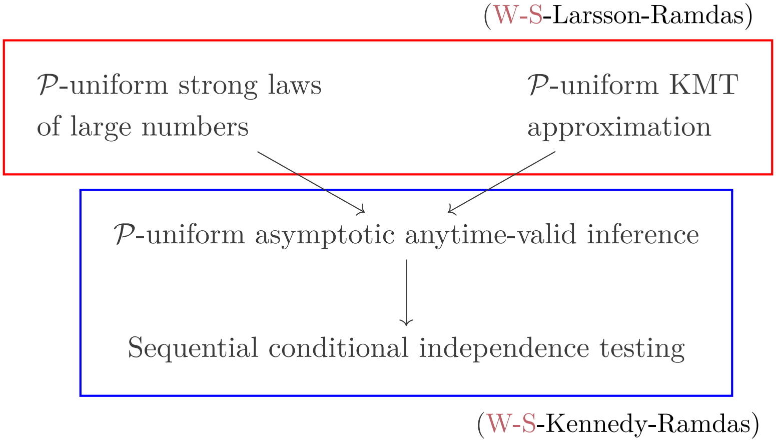

(W-S-Larsson-Ramdas)

$\mathcal{P}$-uniform strong laws of large numbers

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$\mathcal{P}$-uniform asymptotic anytime-valid inference

Sequential conditional independence testing

(W-S-Kennedy-Ramdas)

Thank you

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Supplementary slides

$\mathcal{P}$-uniform anytime-valid inference

Distribution-uniform anytime-valid inference and the Robbins-Siegmund distributions

Ian Waudby-Smith[†], Edward H. Kennedy[‡], and Aaditya Ramdas[‡]

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[‡] Carnegie Mellon University
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Brief detour: **uniform** vs **pointwise** asymptotics

Mathematically, the distinction is quite simple.

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$$P\text{-pointwise:} \quad \sup_{P \in \mathcal{P}} \limsup_{n \to \infty} P\left(\mu \notin C_n\right) \leqslant \alpha.$$

$$\mathcal{P}\text{-uniform:} \quad \limsup_{n \to \infty} \sup_{P \in \mathcal{P}} P\left(\mu \notin C_n\right) \leqslant \alpha.$$

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Many refer to distribution-uniformity as “honesty”.

→ See Li (1989), Kuchibhotla, Balakrishnan, and Wasserman (2022, ’23)

→ Also, Robins et al. (2003) or §1.2.4 in Tsybakov for pointwise “pathologies”.

What is unsettling about pointwise asymptotics?

Thought experiment: Suppose we want type-I error at most $\alpha + \varepsilon$. Take $n \in \{1, 2, \dots\}$ to be *as large as you like.*

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Pointwise: $\exists P^{(n)} \in \mathcal{P}$ so that $P^{(n)}(\mu \notin C_n) > \alpha + \varepsilon$.

Uniform: $\forall P \in \mathcal{P},$ we have $P(\mu \notin C_n) < \alpha + \varepsilon$.

A guarantee like

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} P\left(\mu \notin C_n\right) \leqslant \alpha$$

allows us to articulate what distributional properties certain asymptotic approximations are *insensitive* to.

</detour>

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This will lead to strictly stronger statistical guarantees and motivate some interesting explorations in probability (latter half of the talk).

Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. on $(\Omega, \mathcal{F}, P)_{P\in\mathcal{P}}$ where $\mathcal{P}$ is a collection of distributions for which

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Theorem (W-S et al., 2024b): $\mathcal{P}$-uniform anytime-valid inference.

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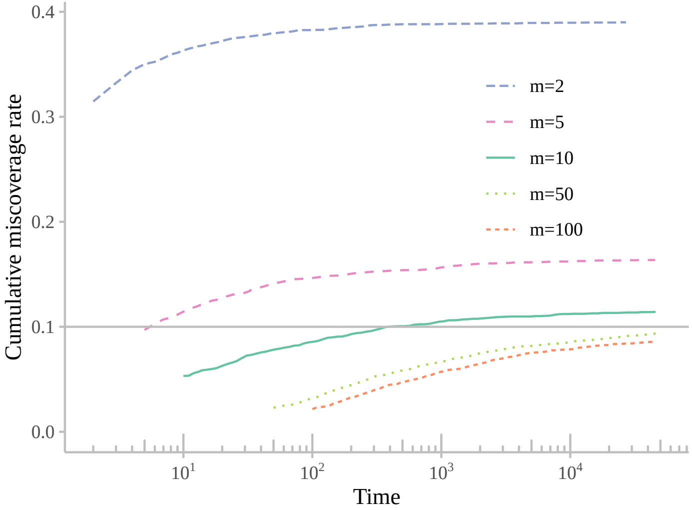
we now have asymptotic $(1-\alpha)$-coverage *uniformly* in $\mathcal{P}$:

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)}\right) = \alpha.$$

Asymptotic anytime-valid inference in a picture



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where Φ is the Gaussian CDF and ϕ is the Gaussian density.

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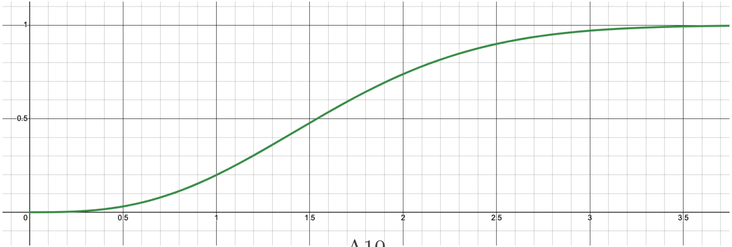

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$$\zeta := \sup_{t\geqslant 1}\left\{\frac{W(t)^2}{t} - \log(t)\right\}.$$

Then $\mathbb{P}(\zeta \leqslant x) = \Psi(x)$; $x \geqslant 0$.
(see Robbins & Siegmund (1970) or W-S et al., 2024a).

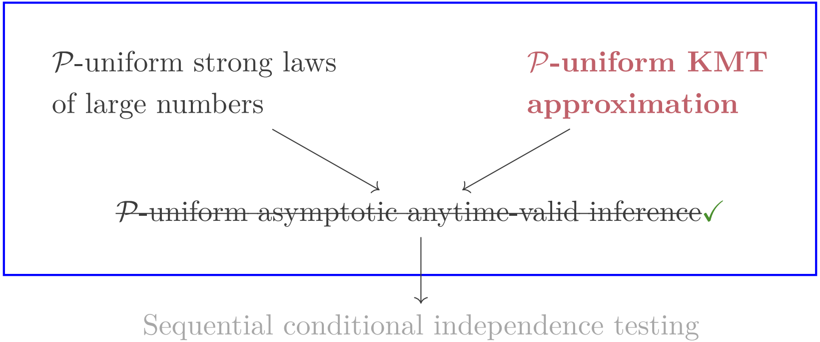
How can we formally relate the distributions of

$$\sup_{k \geqslant m} \left\{ \frac{k \cdot (\widehat{\mu}_k - \mu)^2}{\widehat{\sigma}_k^2} - \log(k/m) \right\} \quad \text{and} \quad \sup_{t \geqslant 1} \left\{ \frac{W(t)^2}{t} - \log(t) \right\}$$

for large m?

The answer relies on some new strong Gaussian approximation theory.

An overview of this line of work



$\mathcal{P}$-uniform strong laws of large numbers

$\mathcal{P}$-uniform KMT approximation

~~$\mathcal{P}$-uniform asymptotic anytime-valid inference~~✓

Sequential conditional independence testing

$\mathcal{P}$-uniform Komlós-Major-Tusnády (KMT) approximation

(Or strong Gaussian approximation; strong invariance principles)

Nonasymptotic and distribution-uniform Komlós-Major-Tusnády approximation

Ian Waudby-Smith[†], Martin Larsson[‡], and Aaditya Ramdas[‡]

[†]University of California, Berkeley
[‡]Carnegie Mellon University

A crash course on strong Gaussian approximation

A crash course on strong Gaussian approximation

Suppose $(X_n)_{n\in\mathbb{N}}$ are mean-zero i.i.d. on $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}_P|X|^q < \infty$; $q > 2$. While the CLT allows us to conclude

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i/\sigma \xrightarrow{d} N(0,1),$$

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more can be said.
There exists an implicit $(Y_n)_{n\in\mathbb{N}}$ of i.i.d. $N(0,\sigma^2)$ such that

$$\left| \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i \right| = o\left(n^{1/q}\right) \quad P\text{-a.s.}$$

(Strassen ’64, ’67; Komlós-Major-Tusnády ’75, ’76, Major ’76).

A crash course on strong Gaussian approximation

A statement like

$$\left| \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i \right| = o\left(n^{1/q}\right) \quad P\text{-a.s.}$$

was used in Bibaut et al. (2022) and W-S et al. (2024a) to derive P-pointwise anytime-valid inference guarantees like

$$\forall P \in \mathcal{P}, \quad \lim_{m \to \infty} P\left(\exists k \geqslant m : \mu \notin C_k^{(m)}\right) = \alpha.$$

However, the results of Strassen, KMT, and the surrounding literature is fundamentally P-pointwise.

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Let us now generalize the notion of strong approximations to classes of distributions $\mathcal{P}$ and show exactly when $o(n^{1/q})$ rates are attainable.

(Our results are actually *nonasymptotic*, but I will omit those details.)

Definition (W-S et al., 2025): $\mathcal{P}$-uniform strong approximation

We say $\sum_{i=1}^{n} X_i$ satisfies a $\mathcal{P}$-uniform strong approximation on $(\Omega, \mathcal{F}, P)_{P \in \mathcal{P}}$ with a rate of r_n if

$$\forall \varepsilon > 0, \quad \lim_{m \to \infty} \sup_{P \in \mathcal{P}} P\left(\sup_{k \geqslant m} \left| \frac{\sum_{i=1}^{k} X_i - \sum_{i=1}^{k} Y_i}{r_k} \right| \geqslant \varepsilon \right) = 0.$$

Shorthand notation:

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Note when $\mathcal{P} = \{P\}$: $\quad \sup_{k \geqslant m} |Z_k| = o_P(1)$ *if and only if* $Z_n = o(1)$ P-a.s.

Theorem (W-S et al., 2025): $\mathcal{P}$-uniform KMT approximation, Part I.

Suppose $(X_n)_{n\in\mathbb{N}}$ are i.i.d. mean-zero and unit variance on $(\Omega, \mathcal{F}, P)_{P\in\mathcal{P}}$. Then

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P\left[|X|^q \mathbb{1}\{|X|^q > m\}\right] = 0$$

if and only if

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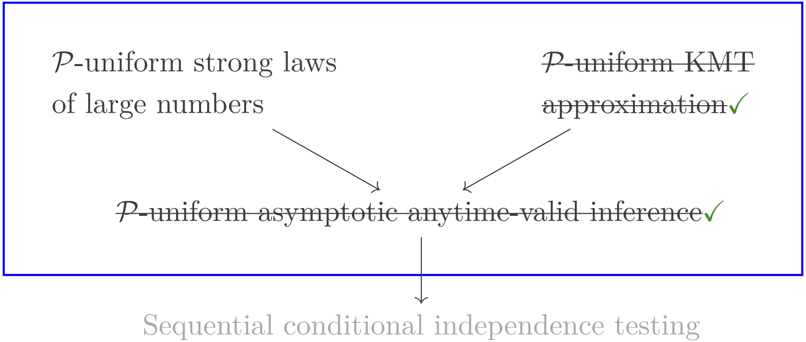
Corollary (KMT '76): $\mathbb{E}_P|X|^q < \infty \iff \sum_{i=1}^n X_i - \sum_{i=1}^n Y_i = o(n^{1/q})$ P-a.s.

An overview of this line of work



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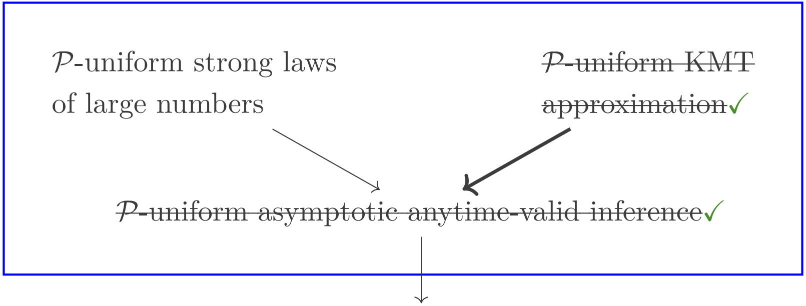
Sequential conditional independence testing

An overview of this line of work



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Sequential conditional independence testing

Returning back to the convergence in distribution result,

$$\sup_{k \geqslant m} \left\{ \text{statistic}_k^{(m)} \right\} \xrightarrow{\mathcal{P}} \Psi(x)$$

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(d) Recall that $\zeta := \sup\limits_{t\geqslant 1} \left\{(W(t))^2/t - \log(t)\right\} \sim \Psi$. ☐

However, there's one last missing piece. The statement

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \sup_{x\geqslant 0} \left| P\left(\sup_{k\geqslant m} \left\{ \frac{k\cdot(\widehat{\mu}_k-\mu_P)^2}{\widehat{\sigma}_k^2} - \log(k/m) \right\} \leqslant x \right) - \Psi(x) \right| = 0$$

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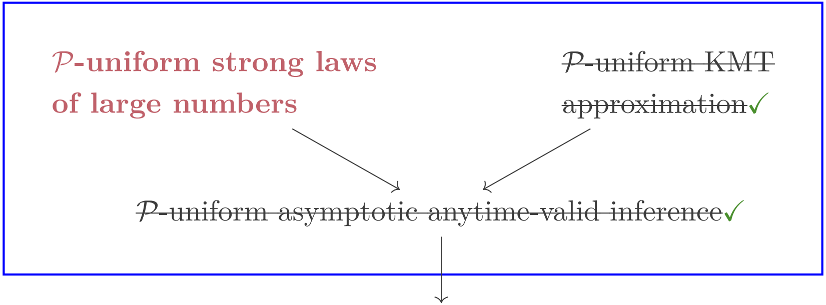
involves $\widehat{\sigma}_k^2$ $\underline{\textit{not}}$ σ_P^2.

In a nutshell, we need that

$$\widehat{\sigma}_n^2 - \sigma_P^2 = o(n^{-\beta})$$

almost surely and $\mathcal{P}$-uniformly for some $\beta > 0$.

An overview of this line of work



$\mathcal{P}$-uniform strong laws of large numbers

~~$\mathcal{P}$-uniform KMT approximation~~✓

~~$\mathcal{P}$-uniform asymptotic anytime-valid inference~~✓

Sequential conditional independence testing

$\mathcal{P}$-uniform strong laws of large numbers

Distribution-uniform strong laws of large numbers

Ian Waudby-Smith[†], Martin Larsson[‡], and Aaditya Ramdas[‡]

[†]University of California, Berkeley
[‡]Carnegie Mellon University

A.N. Kolmogorov's SLLN (1930):

X is integrable $\mathbb{E}_P|X| < \infty$ **if and only if**

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}_P X = o(1) \quad P\text{-a.s.}$$

This is a P-*pointwise* statement.

Kai Lai Chung's SLLN (1951):

*"For use in **statistical** applications, Professor **Wald** raised the question of the uniformity of the SLLN with respect to a family of distributions."*

If X is $\mathcal{P}$-*uniformly integrable*, i.e.

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P \{|X| \mathbb{1}\{|X| > m\}\} = 0,$$

Then

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}X = \bar{o}_{\mathcal{P}}(1).$$

However, both Kolmogorov's and Chung's SLLNs say that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}_P(X)) \to 0 \quad \text{a.s.},$$

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(i) Can anything be said about *how fast* this converges?

(ii) If $\mathbb{E}_P|X| = \infty$ (e.g. Cauchy), can we control how fast this *diverges*?

Marcinkiewicz-Zygmund (**M-Z**) SLLN (1937):

$\mathbb{E}_P|X|^q < \infty$; $q \in [1,2)$ **if and only if**

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}_P X = o\left(n^{1/q-1}\right) \quad P\text{-a.s.}$$

Moreover, for $q \in (0,1)$, $\mathbb{E}_P|X|^q < \infty$ **if and only if**

$$\frac{1}{n}\sum_{i=1}^{n} X_i = o\left(n^{1/q-1}\right) \quad P\text{-a.s.}$$

Like Kolmogorov's SLLN, this is P-pointwise. What about a $\mathcal{P}$-uniform generalization?

Theorem (W-S et al. 2024c): A $\mathcal{P}$-uniform M-Z SLLN.

For $q \in [1, 2)$,

$$\lim_{m \to \infty} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[|X - \mathbb{E}_P X|^q \mathbb{1}\{ |X - \mathbb{E}_P X|^q \geqslant m \} \right] = 0$$

if and only if

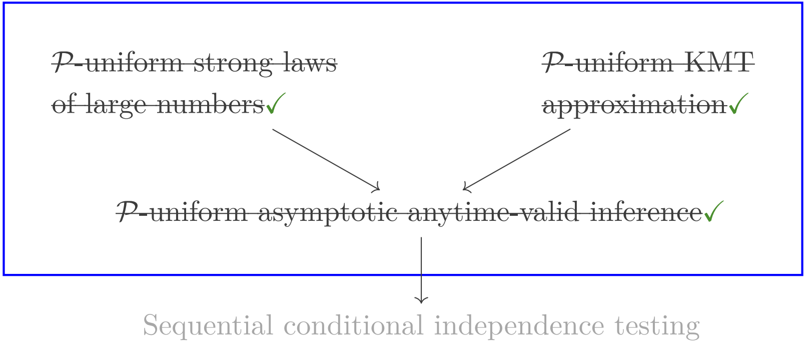
$$\frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}X = \bar{o}_{\mathcal{P}}(n^{1/q-1}),$$

and similarly for $q \in (0, 1)$ but with $\mathbb{E}_P X$ replaced by 0.

Again, unif. integrability $\iff$ $\sup_{P \in \mathcal{P}} \mathbb{E}_P \left[\varphi(|X - \mathbb{E}_P X|^q) \right] < \infty$ for $\varphi(x)/x \to \infty$.

This unifies the SLLNs of Kolmogorov ('30), M-Z ('37), and Chung ('51).

An overview of this line of work



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Sequential conditional independence testing

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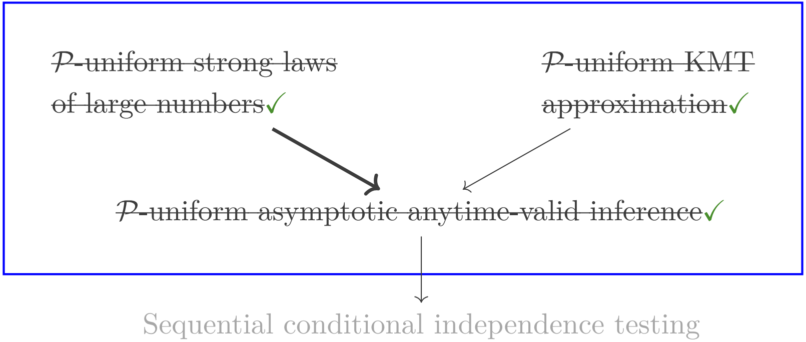

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Sequential conditional independence testing

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we have that $\widehat{\sigma}_k^2 - \sigma_P^2 = \bar{o}_{\mathcal{P}}(n^{-\beta})$ as long as $\sup_{P\in\mathcal{P}} \mathbb{E}_P|X - \mu_P|^{2+\delta} < \infty$, for example.

Theorem (W-S et al., 2024b): $\mathcal{P}$-uniform anytime-valid inference.

Suppose $\sup_{P \in \mathcal{P}} \mathbb{E}_P |X - \mu_P|^{2+\delta} < \infty$.

Then for the sequence of intervals

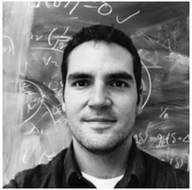
$$C_k^{(m)} := \left[\widehat{\mu}_k \pm \widehat{\sigma}_k \sqrt{\frac{\Psi^{-1}(1-\alpha) + \log(k/m)}{k}} \right],$$

we have

$$\lim_{m \to \infty} \sup_{P \in \mathcal{P}} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)} \right) = \alpha.$$

At the risk of oversimplification, this is a drop-in anytime-valid replacement for CLT-based confidence intervals (and tests, etc.).

Sequential conditional independence testing



Edward H. Kennedy

CMU Statistics

Aaditya Ramdas

CMU Statistics

Given i.i.d. $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$-valued triplets $(X_n, Y_n, Z_n)_{n\in\mathbb{N}}$, we want to test

$$H_0 : X \perp\!\!\!\perp Y \mid Z \quad \text{versus the alternative} \quad H_1 : X \not\!\perp\!\!\!\perp Y \mid Z.$$

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Under the so-called **Model-X** assumption (Candès et al. 2018):

$$\left\{ \textbf{Model-X:} \quad X \mid Z \quad \text{is known exactly} \right\},$$

some very satisfying nonasymptotic anytime-valid tests exist.
(Duan-Ramdas-Wasserman '22, Shaer et al. '23, Grünwald-Henzi-Lardy '23).

What about moving beyond **Model-X**?

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Paraphrasing a quote from Grünwald, Henzi, and Lardy (2023):

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We provide one answer to this open question.

The Hardness of Conditional Independence Testing and the Generalised Covariance Measure

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Jonas Peters†
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March 25, 2022

First, let's discuss their hardness result.

Why do prior works assume that $X \mid Z$ was known (**Model-X**)?

Suppose $(X_i, Y_i, Z_i)_{i=1}^n$ are supported on the unit cube

$$\mathcal{P}^\star = \{\text{dist'ns on } [0,1]^3\}.$$

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Theorem (Shah & Peters '20): Hardness of cond. independence testing

Let $\dot{\Gamma}_n$ be *any* $\mathcal{P}_0^\star$-uniform hypothesis test.

$$\underbrace{\sup_{P \in \mathcal{P}_1^\star} \lim_{n \to \infty} P\left(\dot{\Gamma}_n \text{ rejects}\right)}_{\text{Best-case power}} \leqslant \underbrace{\lim_{n \to \infty} \sup_{P \in \mathcal{P}_0^\star} P\left(\dot{\Gamma}_n \text{ rejects}\right)}_{\text{Worst-case type-I error}} \leqslant \alpha.$$

Why do prior works assume that $X \mid Z$ was known (**Model-X**)?

Suppose $(X_i, Y_i, Z_i)_{i=1}^n$ are supported on the unit cube

$$\mathcal{P}^\star = \{\text{dist'ns on } [0,1]^3\}.$$

We are interested in the null

$$\mathcal{P}_0^\star := \{P \in \mathcal{P}^\star : X \perp\!\!\!\perp_P Y \mid Z\} \quad \text{vs} \quad \mathcal{P}_1^\star := \mathcal{P}^\star \backslash \mathcal{P}_0^\star.$$

Theorem (Shah & Peters '20): Hardness of cond. independence testing

Let $\dot{\Gamma}_n$ be *any* $\mathcal{P}_0^\star$-uniform hypothesis test.

$$\underbrace{\sup_{P \in \mathcal{P}_1^\star} \lim_{n \to \infty} P\left(\dot{\Gamma}_n \text{ rejects}\right)}_{\text{Best-case power}} \leqslant \underbrace{\lim_{n \to \infty} \sup_{P \in \mathcal{P}_0^\star} P\left(\dot{\Gamma}_n \text{ rejects}\right)}_{\text{Worst-case type-I error}} \leqslant \alpha.$$

In words: "*The **most powerful** uniform test is the one that ignores* $(X_i, Y_i, Z_i)_{i=1}^n$ *and rejects with probability* α."

Anytime-valid conditional independence testing is similarly hard.

Theorem (W-S et al., 2024b): Hardness of anytime-valid cond. indep. testing

Let $\bar{\Gamma}_n^{(m)}$ be *any* $\mathcal{P}_0^\star$-uniform anytime test.

$$\underbrace{\sup_{P \in \mathcal{P}_1^\star} \lim_{m \to \infty} P\left(\exists k \geqslant m : \bar{\Gamma}_k^{(m)} \text{ rejects}\right)}_{\text{Best-case anytime power}} \leqslant \underbrace{\lim_{m \to \infty} \sup_{P \in \mathcal{P}_0^\star} P\left(\exists k \geqslant m : \bar{\Gamma}_k^{(m)} \text{ rejects}\right)}_{\text{Worst-case anytime type-I error}} \leqslant \alpha.$$

Despite their pessimistic result, Shah & Peters show that there is still hope.

Despite their pessimistic result, Shah & Peters show that there is still hope. Using $(X_n, Y_n, Z_n)_{n\in\mathbb{N}}$, define

$$R_{i,n} := \{X_i - \widehat{\mu}_n^x(Z_i)\} \cdot \{Y_i - \widehat{\mu}_n^y(Z_i)\},$$

where $\widehat{\mu}_n^x$ and $\widehat{\mu}_n^y$ are estimates of $\mu^x(Z) := \mathbb{E}(X \mid Z)$ and $\mu^y(Z) := \mathbb{E}(Y \mid Z)$.

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where $\widehat{\mu}_n^x$ and $\widehat{\mu}_n^y$ are estimates of $\mu^x(Z) := \mathbb{E}(X \mid Z)$ and $\mu^y(Z) := \mathbb{E}(Y \mid Z)$. Define the *generalized covariance measure* (GCM) statistic:

$$\mathrm{G\dot{C}M}_n := \frac{1}{n}\sum_{i=1}^{n} R_{i,n}/\widehat{\sigma}_n.$$

Intuition: if $\widehat{\mu}_n^x \approx \mu^x$ and $\widehat{\mu}_n^y \approx \mu^y$, then

$$R_{i,n} \approx \{X_i - \mu^x(Z_i)\} \cdot \{Y_i - \mu^y(Z_i))\}, \quad \mathbb{E}_{H_0}(R_{i,n}) = 0.$$

Theorem (Shah & Peters '20): The GCM test

Suppose $\sup_{P \in \mathcal{P}_0} \mathbb{E}_P |\{X_i - \mu^x(Z_i)\} \cdot \{Y_i - \mu^y(Z_i))\}|^{2+\delta} < \infty$ and

$$\sup_{P \in \mathcal{P}_0} \|\widehat{\mu}_n^x - \mu^x\|_{L_2(P)} \cdot \|\widehat{\mu}_n^y - \mu^y\|_{L_2(P)} = o(1/\sqrt{n}).$$

$$\text{Then} \quad \lim_{n \to \infty} \sup_{P \in \mathcal{P}_0} \sup_{\alpha \in (0,1)} |P\left(p_n \leqslant \alpha\right) - \alpha| = 0,$$

where $p_n := 1 - \Phi(\sqrt{n}\dot{\mathrm{GCM}}_n)$.

In particular, $\dot{\Gamma}_n := \mathbb{1}\{|\sqrt{n}\dot{\mathrm{GCM}}| \geqslant \Phi^{-1}(1 - \alpha/2)\}$ is a $\mathcal{P}_0$-uniform level-α test for conditional independence with nontrivial power!

The *sequential* generalized covariance measure (SeqGCM)

Define $\overline{\mathrm{GCM}}_k := \dfrac{1}{k}\displaystyle\sum_{i=1}^{k}\{X_i - \widehat{\mu}_i^x(Z_i)\} \cdot \{Y_i - \widehat{\mu}_i^y(Z_i))\}/\widehat{\sigma}_k.$

The *sequential* generalized covariance measure (SeqGCM)

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Theorem (W-S et al., 2024b): The SeqGCM test

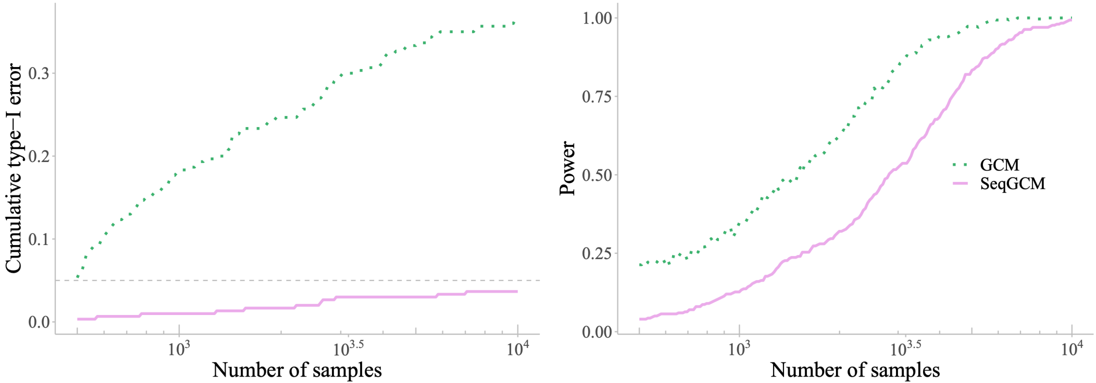
Under the same conditions as the GCM test (Shah & Peters, 2020), suppose

$$\sup_{P \in \mathcal{P}_0} \|\widehat{\mu}_n^x - \mu^x\|_{L_2(P)} \cdot \|\widehat{\mu}_n^y - \mu^y\|_{L_2(P)} = O\left(1/\sqrt{n \log^{2+\delta}(n)}\right).$$

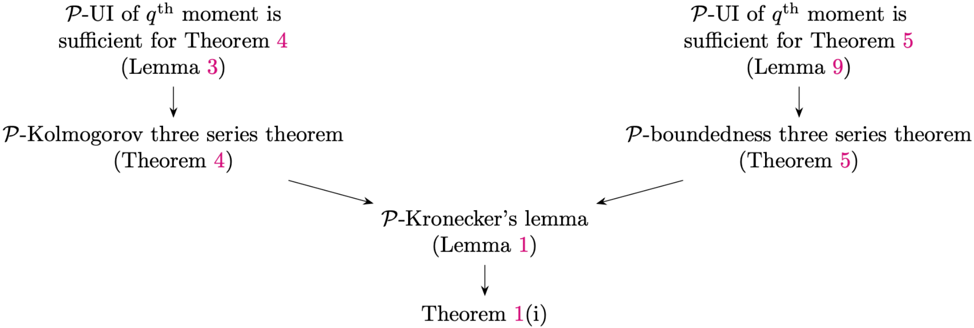
$$\text{Then,} \quad \lim_{m \to \infty} \sup_{P \in \mathcal{P}_0} P\left(\exists k \geqslant m : \overline{p}_k^{(m)} \leqslant \alpha\right) = \alpha,$$

where $\overline{p}_k^{(m)} := 1 - \Psi(k\overline{\mathrm{GCM}}_k^2 - \log(k/m))$.

GCM vs SeqGCM: type-I error and power



	Kolmogorov (1937)	**Chung (1951)**	**M-Z (1937)**	**This work**
$q = 1$	✓	✓	✓	✓
$q \in (0, 2)$			✓	✓
$\mathcal{P}$-uniform		✓		✓
“if and only if”	✓		✓	✓


$\mathcal{P}$-UI of q^{th} moment is sufficient for Theorem 4 (Lemma 3)
$\mathcal{P}$-UI of q^{th} moment is sufficient for Theorem 5 (Lemma 9)
$\mathcal{P}$-Kolmogorov three series theorem (Theorem 4)
$\mathcal{P}$-boundedness three series theorem (Theorem 5)
$\mathcal{P}$-Kronecker's lemma (Lemma 1)
Theorem 1(i)

A word on uniform integrability

Uniform integrability of the q^{th} moment,

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P\left[|X-\mathbb{E}_P X|^q \mathbb{1}\{|X-\mathbb{E}_P X|^q \geqslant m\}\right] = 0$$

is equivalent to the $\varphi(|\cdot|^q)^{\text{th}}$ moment being $\mathcal{P}$-uniformly bounded:

$$\sup_{P\in\mathcal{P}} \mathbb{E}_P\left[\varphi(|X-\mathbb{E}_P X|^q)\right] < \infty,$$

for some $\varphi(\cdot) \geqslant 0$ so that $\lim_{y\to\infty} \frac{\varphi(y)}{y} = \infty.$

(due to Charles Jean de la Vallée-Poussin)

A $\mathcal{P}$-uniform SLLN for non-identically distributed random variables

Theorem (W-S et al., 2024c): A $\mathcal{P}$-uniform SLLN for **non-iid** RVs

Let $(X_n)_{n\in\mathbb{N}}$ be independent RVs on $(\Omega, \mathcal{F}, P)_{P\in\mathcal{P}}$. Suppose

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \sum_{k=m}^{\infty} \frac{\mathbb{E}_P|X_k - \mathbb{E}_P X_k|^q}{a_k^q} = 0$$

for some $a_n \nearrow \infty$ and some $q \in [1, 2]$. Then,

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) = \bar{o}_{\mathcal{P}}(a_n/n),$$

This is a $\mathcal{P}$-uniform generalization of the usual independent SLLN (see §IX of Petrov (1975)).