Anytime-valid inference and uniform central limit theory

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Iowa State University: Statistics Seminar, 2026

A brief crash course on *anytime-valid inference*

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Flip the coin $n = 100$ times.

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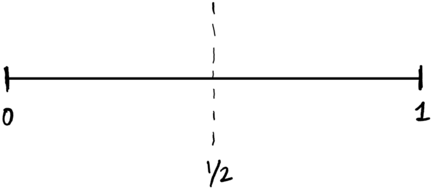

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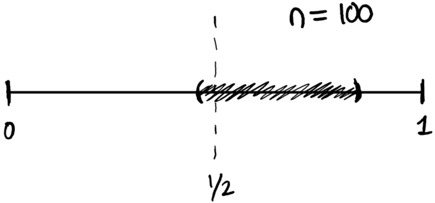

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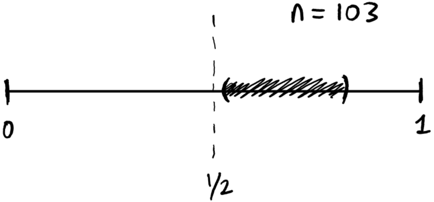

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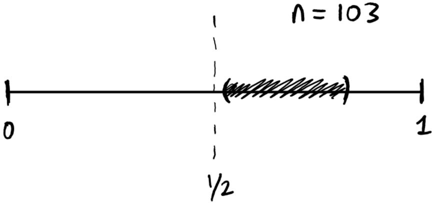
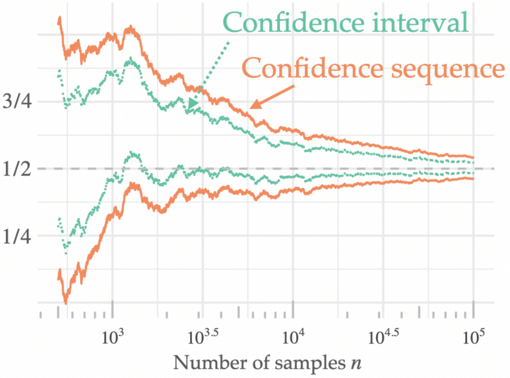


This is tempting, but **inflates type-I error**.

A solution: **Confidence sequences**

Estimation view



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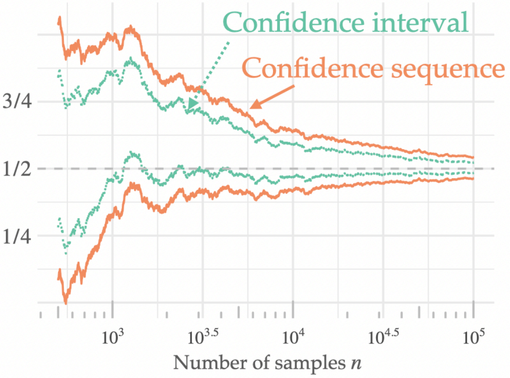

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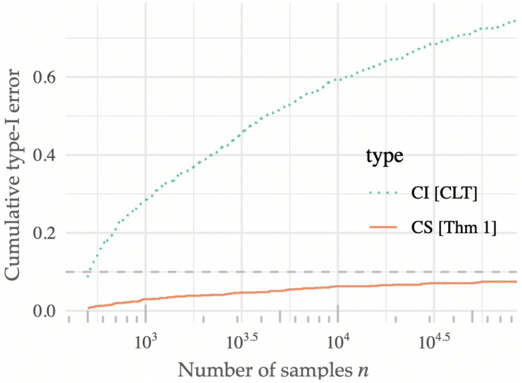

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Confidence sequences, sequential tests, etc. draw on a rich literature:

(i) Ville, Wald ('30–'40s)

(ii) Robbins, Siegmund, Darling, and Lai ('60s–'80s)

(iii) Vovk, Shafer, Johari, Howard, Kaufmann, Ramdas, Wang, Grünwald, de Heide, Koolen, Orabona, Jun, Duchi, and **many** more (2010s-'20s).

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The vast majority of this work has been *nonasymptotic*:

$$\mathbb{P}(\exists n \in \{1, 2, \dots\} : \mathbf{error}_n) \leqslant \alpha.$$

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Non-asymptotic inference is provably impossible	**Central limit theorem**	Today's talk.

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$\mathcal{P}$-uniform strong laws of large numbers

$\mathcal{P}$-uniform KMT approximation

$\mathcal{P}$-uniform asymptotic anytime-valid inference

Sequential conditional independence testing

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Background on *asymptotic* anytime-valid inference

Robbins & Siegmund (1970)

W-S et al., (2024a/2024b)

Bibaut et al. (2022)

Time-uniform central limit theory and asymptotic confidence sequences

Ian Waudby-Smith[1], David Arbour[2], Ritwik Sinha[2], Edward H. Kennedy[1], and Aaditya Ramdas[1]

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edward@stat.cmu.edu, aramdas@cmu.edu

Annals of Statistics, 2024

We will look at *doubly indexed* intervals for each $k \geqslant m$:

$$(C_k^{(1)})_{k=1}^{\infty}, (C_k^{(2)})_{k=2}^{\infty}, (C_k^{(3)})_{k=3}^{\infty}, \dots$$

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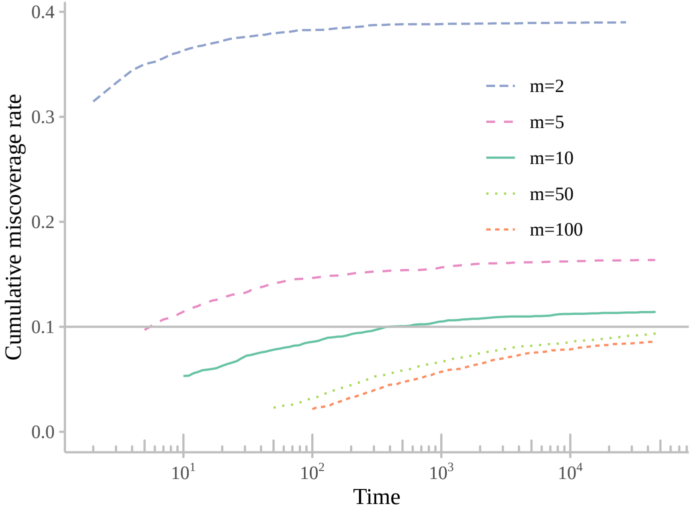
Definition. (W-S et al., AoS 2024a)

We say $C_k^{(m)}$ has asymptotic anytime $(1-\alpha)$-coverage for μ if

$$\forall P \in \mathcal{P}, \quad \limsup_{m \to \infty} P\left(\exists k \geqslant m : \mu \notin C_k^{(m)} \right) \leqslant \alpha.$$

Asymptotic anytime-valid inference in a picture



Example: Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. on $(\Omega, \mathcal{F}, P)$ with mean μ_P and variance σ_P^2.

Define $\widehat{\mu}_n := \frac{1}{n}\sum_{i=1}^n X_i$ and $\widehat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - \widehat{\mu}_n^2$.

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If $\mathbb{E}_P|X|^3 < \infty$, then $(C_k^{(m)})_{k=m}^{\infty}$ has asymptotic coverage for μ_P where

$$C_k^{(m)} := \left[\widehat{\mu}_k \pm \widehat{\sigma}_k \sqrt{\frac{\Psi^{-1}(1-\alpha) + \log(k/m)}{k}}\right].$$

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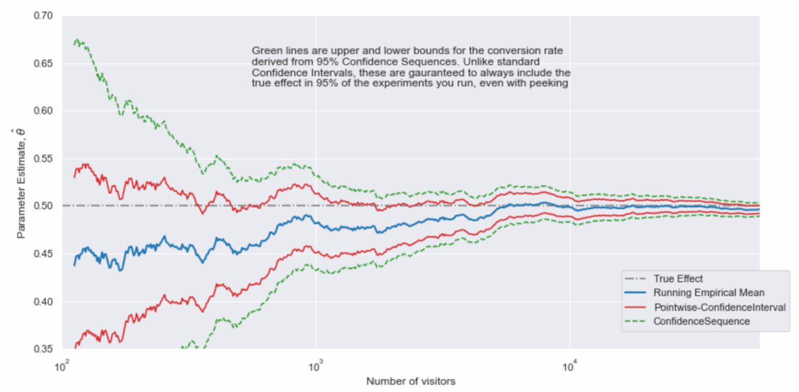
(See W-S et al. and Bibaut et al.). That is,

$$\forall P \in \mathcal{P}, \quad \lim_{m\to\infty} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)}\right) = \alpha.$$

Adobe's Statistical Methodology: Any Time Valid Confidence Sequences

A **Confidence Sequence** is a sequential analog of a **Confidence Interval**, e.g. if you repeat your experiments one hundred times, and calculate an estimate of the mean metric and its associated 95%-Confidence Sequence for every new user that enters the experiment. A 95% Confidence Sequence will include the true value of the metric in 95 out of the 100 experiments that you ran. A 95% Confidence Interval could only be calculated once per experiment in order to give the same 95% coverage guarantee; not with every single new user. Confidence Sequences therefore allow you to continuously monitor experiments, without increasing False Positive error rates.

The difference between confidence sequences and confidence intervals for a single experiment is shown in the animation below:



GrowthBook's implementation

There are many approaches to sequential testing, several of which are well explained and compared in this Spotify blogpost.

For GrowthBook, we selected a method that would work for the wide variety of experimenters that we serve, while also providing experimenters with a way to tune the approach for their setting. To that end, we implement Asymptotic Confidence Sequences introduced by Waudby-Smith et al. (2023); these are very similar to the Generalized Anytime Valid Inference confidence sequences described by Spotify in the above post and introduced by Howard et al. (2022), although the Waudby-Smith et al. approach more transparently applies to our setting.

Specifically, GrowthBook's confidence sequences, which take the place of confidence intervals, become:

$$\left(\hat{\mu} \pm \hat{\sigma} * \sqrt{N} * \sqrt{\frac{2(N\rho^2+1)}{N^2\rho^2} \log\left(\frac{\sqrt{N\rho^2+1}}{\alpha}\right)}\right)$$

$$\rho = \sqrt{\frac{-2\log(\alpha)+\log(-2\log(\alpha)+1)}{N^*}}$$

In the above, $\hat{\mu}$ is our estimated uplift, $\hat{\sigma}$ is the estimated standard error of the uplift, α is our significance level (defaults to 0.05), and N is the sum of the two variation's sample sizes.

However, notice that the guarantee

$$\forall P \in \mathcal{P}, \quad \lim_{m \to \infty} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)}\right) = \alpha$$

is *distribution-pointwise*.

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Many modern asymptotic methods aim to be *distribution-uniform*.

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Distribution-uniform anytime-valid inference and the Robbins-Siegmund distributions

Ian Waudby-Smith[†], Edward H. Kennedy[‡], and Aaditya Ramdas[‡]

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[‡] Carnegie Mellon University
ianws@berkeley.edu {edward,aramdas}@stat.cmu.edu

Brief detour: **uniform** vs **pointwise** asymptotics

Mathematically, the distinction is quite simple.

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$$P\text{-pointwise:} \quad \sup_{P \in \mathcal{P}} \limsup_{n \to \infty} P\left(\mu \notin C_n\right) \leqslant \alpha.$$

$$\mathcal{P}\text{-uniform:} \quad \limsup_{n \to \infty} \sup_{P \in \mathcal{P}} P\left(\mu \notin C_n\right) \leqslant \alpha.$$

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Many refer to distribution-uniformity as "honesty".

→ See Li (1989), Kuchibhotla, Balakrishnan, and Wasserman (2022, '23)

→ Also, Robins et al. (2003) or §1.2.4 in Tsybakov for pointwise "pathologies".

What is unsettling about pointwise asymptotics?

Thought experiment: Suppose we want type-I error at most $\alpha + \varepsilon$. Take $n \in \{1, 2, \dots\}$ to be *as large as you like.*

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Pointwise: $\exists P^{(n)} \in \mathcal{P}$ so that $P^{(n)}(\mu \notin C_n) > \alpha + \varepsilon$.

Uniform: $\forall P \in \mathcal{P}$, we have $P(\mu \notin C_n) < \alpha + \varepsilon$.

A guarantee like

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}} P\left(\mu \notin C_n\right) \leqslant \alpha$$

allows us to articulate what distributional properties certain asymptotic approximations are *insensitive* to.

</detour>

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This will lead to strictly stronger statistical guarantees and motivate some interesting explorations in probability (latter half of the talk).

Let $(X_n)_{n\in\mathbb{N}}$ be i.i.d. on $(\Omega, \mathcal{F}, P)_{P\in\mathcal{P}}$ where $\mathcal{P}$ is a collection of distributions for which

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Theorem (W-S et al., 2024b): $\mathcal{P}$-uniform anytime-valid inference.

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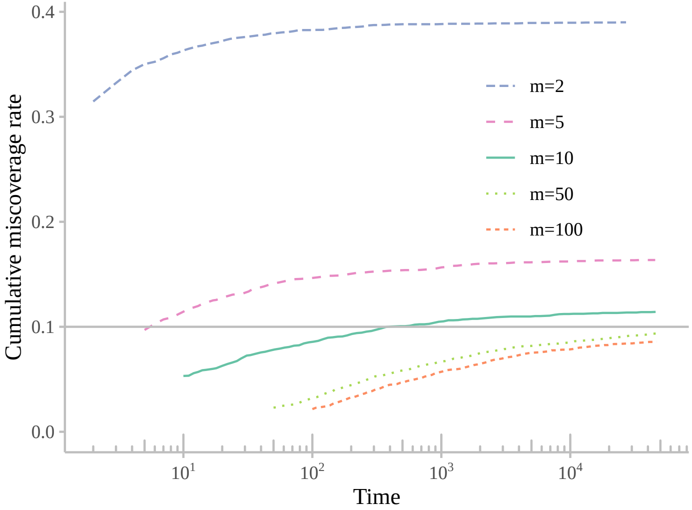
we now have asymptotic $(1-\alpha)$-coverage *uniformly* in $\mathcal{P}$:

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)} \right) = \alpha.$$

Asymptotic anytime-valid inference in a picture



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After some algebraic juggling, this can be reduced to showing that

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \sup_{x\geqslant 0} \left| P\left(\sup_{k\geqslant m} \left\{ \frac{k\cdot(\widehat{\mu}_k - \mu_P)^2}{\widehat{\sigma}_k^2} - \log(k/m) \right\} \leqslant x \right) - \Psi(x) \right| = 0.$$

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→ reduces time- and $\mathcal{P}$-uniform coverage / type-I error control to $\mathcal{P}$-*uniform convergence in distribution* to this d.f. Ψ.

Ψ: The Robbins-Siegmund distribution

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Consider the function

$$\Psi(x) := 1 - 2\left[1 - \Phi(\sqrt{x}) + \sqrt{x}\phi(\sqrt{x})\right];\ x \geqslant 0,$$

where Φ is the Gaussian CDF and ϕ is the Gaussian density.

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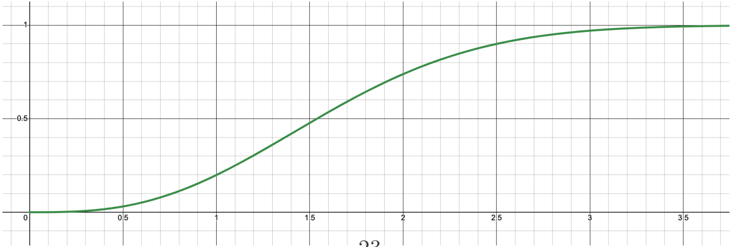

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Take the following transformation:

$$\zeta := \sup_{t \geqslant 1} \left\{ \frac{W(t)^2}{t} - \log(t) \right\}.$$

Then $\mathbb{P}(\zeta \leqslant x) = \Psi(x)$; $x \geqslant 0$.
(see Robbins & Siegmund (1970) or W-S et al., 2024a).

How can we formally relate the distributions of

$$\sup_{k \geqslant m} \left\{ \frac{k \cdot (\widehat{\mu}_k - \mu)^2}{\widehat{\sigma}_k^2} - \log(k/m) \right\} \quad \text{and} \quad \sup_{t \geqslant 1} \left\{ \frac{W(t)^2}{t} - \log(t) \right\}$$

for large m?

The answer relies on some new strong Gaussian approximation theory.

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~~$\mathcal{P}$-uniform asymptotic anytime-valid inference~~✓

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$\mathcal{P}$-uniform Komlós-Major-Tusnády (KMT) approximation

(Or strong Gaussian approximation; strong invariance principles)

Nonasymptotic and distribution-uniform Komlós-Major-Tusnády approximation

Ian Waudby-Smith[†], Martin Larsson[‡], and Aaditya Ramdas[‡]

[†]University of California, Berkeley
[‡]Carnegie Mellon University

A crash course on strong Gaussian approximation

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Suppose $(X_n)_{n\in\mathbb{N}}$ are mean-zero i.i.d. on $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}_P|X|^q < \infty$; $q > 2$. While the CLT allows us to conclude

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i/\sigma \xrightarrow{d} N(0,1),$$

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more can be said.
There exists an implicit $(Y_n)_{n\in\mathbb{N}}$ of i.i.d. $N(0,\sigma^2)$ such that

$$\left| \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i \right| = o\left(n^{1/q}\right) \quad P\text{-a.s.}$$

(Strassen '64, '67; Komlós-Major-Tusnády '75, '76, Major '76).

A crash course on strong Gaussian approximation

A statement like

$$\left| \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} Y_i \right| = o\left(n^{1/q}\right) \quad P\text{-a.s.}$$

was used in Bibaut et al. (2022) and W-S et al. (2024a) to derive
P-pointwise anytime-valid inference guarantees like

$$\forall P \in \mathcal{P}, \quad \lim_{m \to \infty} P\left(\exists k \geqslant m : \mu \notin C_k^{(m)}\right) = \alpha.$$

However, the results of Strassen, KMT, and the surrounding literature is fundamentally P-pointwise.

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Let us now generalize the notion of strong approximations to classes of distributions $\mathcal{P}$ and show exactly when $o(n^{1/q})$ rates are attainable.

(Our results are actually *nonasymptotic*, but I will omit those details.)

Definition (W-S et al., 2025): $\mathcal{P}$-uniform strong approximation

We say $\sum_{i=1}^{n} X_i$ satisfies a $\mathcal{P}$-uniform strong approximation on $(\Omega, \mathcal{F}, P)_{P\in\mathcal{P}}$ with a rate of r_n if

$$\forall \varepsilon > 0, \quad \lim_{m\to\infty} \sup_{P\in\mathcal{P}} P\left(\sup_{k\geqslant m} \left| \frac{\sum_{i=1}^{k} X_i - \sum_{i=1}^{k} Y_i}{r_k} \right| \geqslant \varepsilon \right) = 0.$$

Shorthand notation:

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Note when $\mathcal{P} = \{P\}$: $\sup_{k\geqslant m} |Z_k| = o_P(1)$ *if and only if* $Z_n = o(1)$ P-a.s.

Theorem (W-S et al., 2025): $\mathcal{P}$-uniform KMT approximation, Part I.

Suppose $(X_n)_{n\in\mathbb{N}}$ are i.i.d. mean-zero and unit variance on $(\Omega, \mathcal{F}, P)_{P\in\mathcal{P}}$. Then

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P\left[|X|^q \mathbb{1}\{|X|^q > m\}\right] = 0$$

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Corollary (KMT '76): $\mathbb{E}_P|X|^q < \infty \iff \sum_{i=1}^n X_i - \sum_{i=1}^n Y_i = o(n^{1/q})$ P-a.s.

An overview of this line of work

$\mathcal{P}$-uniform strong laws of large numbers

~~$\mathcal{P}$-uniform KMT approximation~~ ✓

~~$\mathcal{P}$-uniform asymptotic anytime-valid inference~~ ✓

Sequential conditional independence testing

An overview of this line of work

$\mathcal{P}$-uniform strong laws of large numbers

~~$\mathcal{P}$-uniform KMT approximation~~ ✓

~~$\mathcal{P}$-uniform asymptotic anytime-valid inference~~ ✓

Sequential conditional independence testing

Returning back to the convergence in distribution result,

$$\sup_{k \geqslant m} \left\{ \text{statistic}_k^{(m)} \right\} \xrightarrow{\mathcal{P}} \Psi(x)$$

$\square$

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(d) Recall that $\zeta := \sup_{t\geqslant 1} \left\{(W(t))^2/t - \log(t)\right\} \sim \Psi$. □

However, there's one last missing piece. The statement

$$\lim_{m \to \infty} \sup_{P \in \mathcal{P}} \sup_{x \geqslant 0} \left| P \left(\sup_{k \geqslant m} \left\{ \frac{k \cdot (\widehat{\mu}_k - \mu_P)^2}{\widehat{\sigma}_k^2} - \log(k/m) \right\} \leqslant x \right) - \Psi(x) \right| = 0$$

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involves $\widehat{\sigma}_k^2$ <u>*not*</u> σ_P^2.

In a nutshell, we need that

$$\widehat{\sigma}_n^2 - \sigma_P^2 = o(n^{-\beta})$$

almost surely and $\mathcal{P}$-uniformly for some $\beta > 0$.

An overview of this line of work

$\mathcal{P}$-uniform strong laws of large numbers

~~$\mathcal{P}$-uniform KMT approximation~~✓

~~$\mathcal{P}$-uniform asymptotic anytime-valid inference~~✓

Sequential conditional independence testing

$\mathcal{P}$-uniform strong laws of large numbers

Distribution-uniform strong laws of large numbers

Ian Waudby-Smith[†], Martin Larsson[‡], and Aaditya Ramdas[‡]

[†]University of California, Berkeley
[‡]Carnegie Mellon University

A.N. Kolmogorov's SLLN (1930):

X is integrable $\mathbb{E}_P|X| < \infty$ **if and only if**

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}_P X = o(1) \quad P\text{-a.s.}$$

This is a P-*pointwise* statement.

Kai Lai Chung's SLLN (1951):

*"For use in **statistical** applications, Professor **Wald** raised the question of the uniformity of the SLLN with respect to a family of distributions."*

If X is $\mathcal{P}$-*uniformly integrable*, i.e.

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P \{|X| \mathbb{1}\{|X| > m\}\} = 0,$$

Then

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}X = \bar{o}_{\mathcal{P}}(1).$$

However, both Kolmogorov's and Chung's SLLNs say that

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}_P(X)) \to 0 \quad \text{a.s.},$$

whether P-**pointwise** or $\mathcal{P}$-**uniformly**.

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(i) Can anything be said about *how fast* this converges?

(ii) If $\mathbb{E}_P|X| = \infty$ (e.g. Cauchy), can we control how fast this *diverges*?

Marcinkiewicz-Zygmund (**M-Z**) SLLN (1937):

$\mathbb{E}_P|X|^q < \infty$; $q \in [1, 2)$ **if and only if**

$$\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}_P X = o\left(n^{1/q-1}\right) \quad P\text{-a.s.}$$

Moreover, for $q \in (0, 1)$, $\mathbb{E}_P|X|^q < \infty$ **if and only if**

$$\frac{1}{n}\sum_{i=1}^{n} X_i = o\left(n^{1/q-1}\right) \quad P\text{-a.s.}$$

Like Kolmogorov's SLLN, this is P-pointwise. What about a $\mathcal{P}$-uniform generalization?

Theorem (W-S et al. 2024c): A $\mathcal{P}$-uniform M-Z SLLN.

For $q \in [1, 2)$,

$$\lim_{m \to \infty} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[|X - \mathbb{E}_P X|^q \mathbb{1}\{|X - \mathbb{E}_P X|^q \geqslant m\} \right] = 0$$

if and only if

$$\frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}X = \bar{o}_{\mathcal{P}}(n^{1/q-1}),$$

and similarly for $q \in (0, 1)$ but with $\mathbb{E}_P X$ replaced by 0.

Again, unif. integrability $\iff \sup_{P \in \mathcal{P}} \mathbb{E}_P [\varphi(|X - \mathbb{E}_P X|^q)] < \infty$ for $\varphi(x)/x \to \infty$.

This unifies the SLLNs of Kolmogorov ('30), M-Z ('37), and Chung ('51).

An overview of this line of work

~~$\mathcal{P}$-uniform strong laws of large numbers~~ ✓

~~$\mathcal{P}$-uniform KMT approximation~~ ✓

↓

~~$\mathcal{P}$-uniform asymptotic anytime-valid inference~~ ✓

↓

Sequential conditional independence testing

An overview of this line of work

~~$\mathcal{P}$-uniform strong laws of large numbers~~✓

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Sequential conditional independence testing

Returning back to the convergence in distribution result,

$$\lim_{m \to \infty} \sup_{P \in \mathcal{P}} \sup_{x \geqslant 0} \left| P\left(\sup_{k \geqslant m} \left\{ \frac{k \cdot (\widehat{\mu}_k - \mu_P)^2}{\widehat{\sigma}_k^2} - \log(k/m) \right\} \leqslant x \right) - \Psi(x) \right| = 0$$

we have that $\widehat{\sigma}_k^2 - \sigma_P^2 = \bar{o}_{\mathcal{P}}(n^{-\beta})$ as long as $\sup_{P \in \mathcal{P}} \mathbb{E}_P |X - \mu_P|^{2+\delta} < \infty$, for example.

Theorem (W-S et al., 2024b): $\mathcal{P}$-uniform anytime-valid inference.

Suppose $\sup_{P \in \mathcal{P}} \mathbb{E}_P |X - \mu_P|^{2+\delta} < \infty$.

Then for the sequence of intervals

$$C_k^{(m)} := \left[\widehat{\mu}_k \pm \widehat{\sigma}_k \sqrt{\frac{\Psi^{-1}(1-\alpha) + \log(k/m)}{k}} \right],$$

we have

$$\lim_{m \to \infty} \sup_{P \in \mathcal{P}} P\left(\exists k \geqslant m : \mu_P \notin C_k^{(m)} \right) = \alpha.$$

At the risk of oversimplification, this is a drop-in anytime-valid replacement for CLT-based confidence intervals (and tests, etc.).

An overview of this line of work

~~$\mathcal{P}$-uniform strong laws of large numbers~~

~~$\mathcal{P}$-uniform KMT approximation~~

~~$\mathcal{P}$-uniform asymptotic anytime-valid inference~~

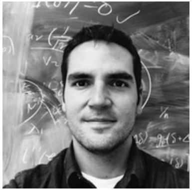
Sequential conditional independence testing

Thank you

ianws.com

Supplementary slides

Sequential conditional independence testing



Edward H. Kennedy

CMU Statistics

Aaditya Ramdas

CMU Statistics

Given i.i.d. $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$-valued triplets $(X_n, Y_n, Z_n)_{n \in \mathbb{N}}$, we want to test

$$H_0 : X \perp\!\!\!\perp Y \mid Z \quad \text{versus the alternative} \quad H_1 : X \not\!\perp\!\!\!\perp Y \mid Z.$$

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Under the so-called **Model-X** assumption (Candès et al. 2018):

$$\left\{ \textbf{Model-X:} \quad X \mid Z \quad \text{is known exactly} \right\},$$

some very satisfying nonasymptotic anytime-valid tests exist.
(Duan-Ramdas-Wasserman '22, Shaer et al. '23, Grünwald-Henzi-Lardy '23).

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Paraphrasing a quote from Grünwald, Henzi, and Lardy (2023):

> *"the anytime-valid tests in this paper are highly tailored to the Model-X assumption, and it is an open question to us as to how to construct general sequential tests of conditional independence without it."*

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We provide one answer to this open question.

The Hardness of Conditional Independence Testing and the Generalised Covariance Measure

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March 25, 2022

First, let's discuss their hardness result.

Why do prior works assume that $X \mid Z$ was known (**Model-X**)?

Suppose $(X_i, Y_i, Z_i)_{i=1}^n$ are supported on the unit cube

$$\mathcal{P}^\star = \{\text{dist'ns on } [0,1]^3\}.$$

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We are interested in the null

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Theorem (Shah & Peters '20): Hardness of cond. independence testing

Let $\dot{\Gamma}_n$ be *any* $\mathcal{P}_0^\star$-uniform hypothesis test.

$$\underbrace{\sup_{P \in \mathcal{P}_1^\star} \lim_{n \to \infty} P\left(\dot{\Gamma}_n \text{ rejects}\right)}_{\text{Best-case power}} \leqslant \underbrace{\lim_{n \to \infty} \sup_{P \in \mathcal{P}_0^\star} P\left(\dot{\Gamma}_n \text{ rejects}\right)}_{\text{Worst-case type-I error}} \leqslant \alpha.$$

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In words: "*The* ***most powerful*** *uniform test is the one that ignores* $(X_i, Y_i, Z_i)_{i=1}^n$ *and rejects with probability* α."

Anytime-valid conditional independence testing is similarly hard.

Theorem (W-S et al., 2024b): Hardness of anytime-valid cond. indep. testing

Let $\bar{\Gamma}_n^{(m)}$ be *any* $\mathcal{P}_0^\star$-uniform anytime test.

$$\underbrace{\sup_{P \in \mathcal{P}_1^\star} \lim_{m \to \infty} P\left(\exists k \geqslant m : \bar{\Gamma}_k^{(m)} \text{ rejects}\right)}_{\text{Best-case anytime power}} \leqslant \underbrace{\lim_{m \to \infty} \sup_{P \in \mathcal{P}_0^\star} P\left(\exists k \geqslant m : \bar{\Gamma}_k^{(m)} \text{ rejects}\right)}_{\text{Worst-case anytime type-I error}} \leqslant \alpha.$$

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Despite their pessimistic result, Shah & Peters show that there is still hope. Using $(X_n, Y_n, Z_n)_{n\in\mathbb{N}}$, define

$$R_{i,n} := \{X_i - \widehat{\mu}_n^x(Z_i)\} \cdot \{Y_i - \widehat{\mu}_n^y(Z_i)\},$$

where $\widehat{\mu}_n^x$ and $\widehat{\mu}_n^y$ are estimates of $\mu^x(Z) := \mathbb{E}(X \mid Z)$ and $\mu^y(Z) := \mathbb{E}(Y \mid Z)$.

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where $\widehat{\mu}_n^x$ and $\widehat{\mu}_n^y$ are estimates of $\mu^x(Z) := \mathbb{E}(X \mid Z)$ and $\mu^y(Z) := \mathbb{E}(Y \mid Z)$. Define the *generalized covariance measure* (GCM) statistic:

$$\dot{\mathrm{GCM}}_n := \frac{1}{n}\sum_{i=1}^{n} R_{i,n}/\widehat{\sigma}_n.$$

Intuition: if $\widehat{\mu}_n^x \approx \mu^x$ and $\widehat{\mu}_n^y \approx \mu^y$, then

$$R_{i,n} \approx \{X_i - \mu^x(Z_i)\} \cdot \{Y_i - \mu^y(Z_i))\}, \quad \mathbb{E}_{H_0}(R_{i,n}) = 0.$$

Theorem (Shah & Peters '20): The GCM test

Suppose $\sup_{P \in \mathcal{P}_0} \mathbb{E}_P |\{X_i - \mu^x(Z_i)\} \cdot \{Y_i - \mu^y(Z_i))\}|^{2+\delta} < \infty$ and

$$\sup_{P \in \mathcal{P}_0} \|\widehat{\mu}_n^x - \mu^x\|_{L_2(P)} \cdot \|\widehat{\mu}_n^y - \mu^y\|_{L_2(P)} = o(1/\sqrt{n}).$$

$$\text{Then} \quad \lim_{n \to \infty} \sup_{P \in \mathcal{P}_0} \sup_{\alpha \in (0,1)} |P(p_n \leqslant \alpha) - \alpha| = 0,$$

where $p_n := 1 - \Phi(\sqrt{n}\dot{\text{GCM}}_n)$.

In particular, $\dot{\Gamma}_n := \mathbb{1}\{|\sqrt{n}\dot{\text{GCM}}| \geqslant \Phi^{-1}(1 - \alpha/2)\}$ is a $\mathcal{P}_0$-uniform level-α test for conditional independence with nontrivial power!

The *sequential* generalized covariance measure (SeqGCM)

Define $\overline{\mathrm{GCM}}_k := \frac{1}{k}\sum_{i=1}^{k}\{X_i - \widehat{\mu}_i^x(Z_i)\} \cdot \{Y_i - \widehat{\mu}_i^y(Z_i))\}/\widehat{\sigma}_k$.

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Theorem (W-S et al., 2024b): The SeqGCM test

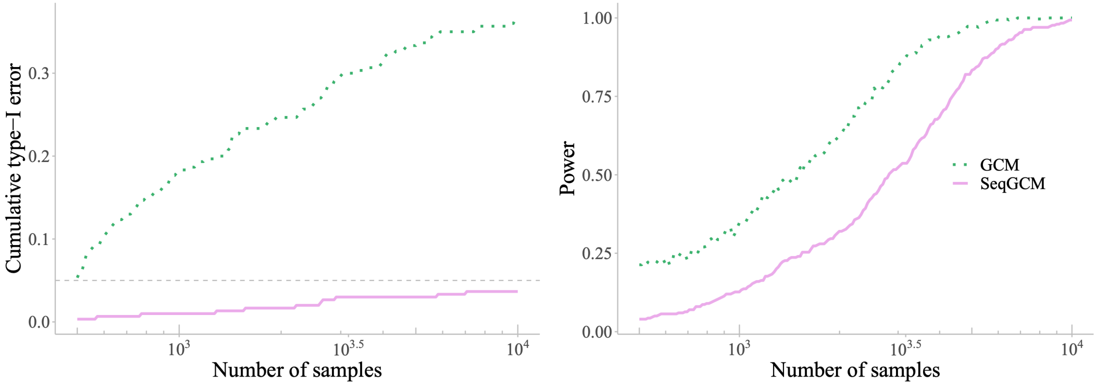
Under the same conditions as the GCM test (Shah & Peters, 2020), suppose

$$\sup_{P \in \mathcal{P}_0} \|\widehat{\mu}_n^x - \mu^x\|_{L_2(P)} \cdot \|\widehat{\mu}_n^y - \mu^y\|_{L_2(P)} = O\left(1/\sqrt{n \log^{2+\delta}(n)}\right).$$

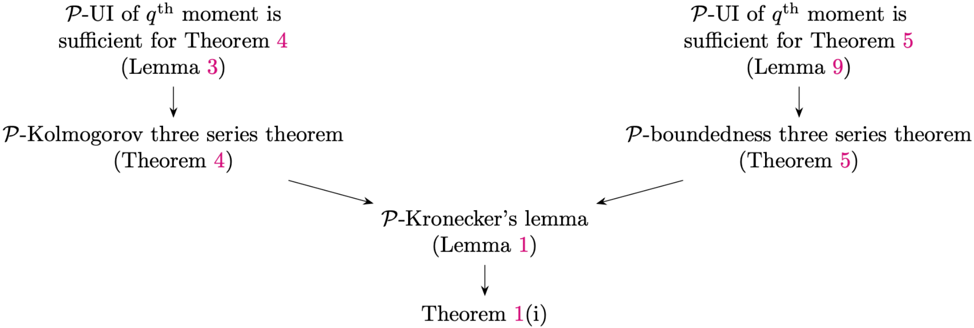
$$\text{Then,} \quad \lim_{m\to\infty} \sup_{P \in \mathcal{P}_0} P\left(\exists k \geqslant m : \bar{p}_k^{(m)} \leqslant \alpha\right) = \alpha,$$

where $\bar{p}_k^{(m)} := 1 - \Psi(k\overline{\text{GCM}}_k^2 - \log(k/m))$.

GCM vs SeqGCM: type-I error and power



	Kolmogorov (1937)	**Chung (1951)**	**M-Z (1937)**	**This work**
$q = 1$	✓	✓	✓	✓
$q \in (0, 2)$			✓	✓
$\mathcal{P}$-uniform		✓		✓
“if and only if”	✓		✓	✓



$\mathcal{P}$-UI of q^{th} moment is sufficient for Theorem 4 (Lemma 3)
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$\mathcal{P}$-Kolmogorov three series theorem (Theorem 4)
$\mathcal{P}$-boundedness three series theorem (Theorem 5)
$\mathcal{P}$-Kronecker's lemma (Lemma 1)
Theorem 1(i)

A word on uniform integrability

Uniform integrability of the q^{th} moment,

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \mathbb{E}_P \left[|X - \mathbb{E}_P X|^q \mathbb{1}\{|X - \mathbb{E}_P X|^q \geqslant m\}\right] = 0$$

is equivalent to the $\varphi(|\cdot|^q)^{\text{th}}$ moment being $\mathcal{P}$-uniformly bounded:

$$\sup_{P\in\mathcal{P}} \mathbb{E}_P \left[\varphi(|X - \mathbb{E}_P X|^q)\right] < \infty,$$

for some $\varphi(\cdot) \geqslant 0$ so that $\lim_{y\to\infty} \frac{\varphi(y)}{y} = \infty.$

(due to Charles Jean de la Vallée-Poussin)

A $\mathcal{P}$-uniform SLLN for non-identically distributed random variables

Theorem (W-S et al., 2024c): A $\mathcal{P}$-uniform SLLN for **non-iid** RVs

Let $(X_n)_{n\in\mathbb{N}}$ be independent RVs on $(\Omega, \mathcal{F}, P)_{P\in\mathcal{P}}$. Suppose

$$\lim_{m\to\infty} \sup_{P\in\mathcal{P}} \sum_{k=m}^{\infty} \frac{\mathbb{E}_P|X_k - \mathbb{E}_P X_k|^q}{a_k^q} = 0$$

for some $a_n \nearrow \infty$ and some $q \in [1, 2]$. Then,

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) = \bar{o}_{\mathcal{P}}(a_n/n),$$

This is a $\mathcal{P}$-uniform generalization of the usual independent SLLN (see §IX of Petrov (1975)).